

HODGE FILTERED COMPLEX BORDISM

MICHAEL J. HOPKINS AND GEREON QUICK

ABSTRACT. We construct Hodge filtered cohomology groups for complex manifolds that combine the topological information of generalized cohomology theories with geometric data of Hodge filtered holomorphic forms. This theory provides a natural generalization of Deligne cohomology. For smooth complex algebraic varieties, we show that the theory satisfies a projective bundle formula and A^1 -homotopy invariance. Moreover, we obtain transfer maps along projective morphisms.

1. INTRODUCTION

For a complex manifold, Deligne cohomology is an elegant way to combine the topological information of integral cohomology with the geometric data of holomorphic forms. For a given p , the Deligne cohomology group $H_{\mathcal{D}}^p(X; \mathbb{Z}(p))$ of a complex manifold X is defined to be the n th hypercohomology of the Deligne complex $\mathbb{Z}_{\mathcal{D}}(p)$

$$0 \rightarrow \mathbb{Z} \xrightarrow{(2\pi i)^p} \mathcal{O}_X \rightarrow \Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_X^{p-1} \rightarrow 0$$

of sheaves over X , where Ω_X^k denotes the sheaf of holomorphic k -forms.

Deligne cohomology is an important tool for many fundamental questions in algebraic, arithmetic and complex geometry. Let us just highlight one of the algebraic geometric applications of Deligne cohomology. Let X be a smooth projective complex variety. The associated complex manifold is an example of a compact Kähler manifold. Let $CH^p X$ be the Chow group of codimension p algebraic cycles modulo rational equivalence. There is the classical cycle map

$$\mathrm{cl}_H : CH^p X \rightarrow H^{2p}(X; \mathbb{Z})$$

from Chow groups to the integral cohomology of the associated complex manifold of X . This map sends a smooth irreducible subvariety $Z \subset X$ of codimension p in X to the Poincaré dual of the fundamental class of Z . The so defined cohomology class of Z is an integral Hodge class, i.e. lies in the group

$$\mathrm{Hdg}^{2p}(X) := H^{2p}(X; \mathbb{Z}) \cap H^{p,p}(X; \mathbb{C})$$

of the Hodge decomposition.

Now Deligne cohomology groups enter the picture. Deligne showed that for every given p the group $H_{\mathcal{D}}^{2p}(X; \mathbb{Z}(p))$ sits in a short exact sequence

$$(1) \quad 0 \rightarrow J^{2p-1}(X) \rightarrow H_{\mathcal{D}}^{2p}(X; \mathbb{Z}(p)) \rightarrow \mathrm{Hdg}^{2p}(X) \rightarrow 0$$

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where the group on the left hand side is the p th intermediate Jacobian of X . Moreover, Deligne constructed a cycle map

$$\mathrm{cl}_{H_{\mathcal{D}}} : CH^p X \rightarrow H_{\mathcal{D}}^{2p}(X; \mathbb{Z}(p)).$$

Together with the short exact sequence (1) this provided an algebraic definition of the classical Abel-Jacobi homomorphism from cycles homologically equivalent to zero, i.e. those which are mapped to zero under cl_H , to the intermediate Jacobian $J^{2p-1}(X)$ (see [11]). So Deligne groups play an important role in the study of the cycle map, the Abel-Jacobi map and, in particular, the Griffiths group of smooth projective complex varieties, i.e. the quotient group of cycles homologically equivalent to zero modulo a potentially weaker relation, called algebraic equivalence.

More recently Totaro showed that the Griffiths group of a complex variety can be analyzed using the purely topological information of its complex cobordism groups. In [30] he proved that the map cl_H factors through a cycle map

$$\mathrm{cl}_{MU} : CH^p X \rightarrow MU^{2p}(X) \otimes_{MU^*} \mathbb{Z}$$

for any smooth projective complex variety X ; in fact, Totaro showed a more general result for any complex variety in terms of the corresponding bordism quotient groups. Since the canonical map $\theta : MU^{2p}(X) \otimes_{MU^*} \mathbb{Q} \rightarrow H^{2p}(X; \mathbb{Q})$ is an isomorphism after tensoring with \mathbb{Q} , this factorization is a torsion phenomenon. But in general, the kernel of θ can be nontrivial. Generalizing work of Atiyah and Hirzebruch, Totaro constructed via a beautiful computation elements in the kernel of θ which are given by Chern classes of suitable representations of a finite group and lie in the image of cl_{MU} . Thereby he found new examples of nontrivial elements in the Griffiths group of Godeaux-Serre varieties. In [25], the second author extended Totaro's idea for the cycle map to varieties over algebraically closed fields of positive characteristic.

Now given that Deligne cohomology and complex cobordism are useful tools for the understanding of the cycle map and also for the Abel-Jacobi map, the question arises whether it is possible to combine these two approaches. Thereby asking whether it is possible to combine the geometric information of differential forms on a complex manifold with the topological information of complex cobordism groups which carry richer data than cohomology groups. The purpose of this paper is to give a positive answer to this question by constructing a natural generalization of Deligne cohomology via complex cobordism and to set the stage for its applications, in particular with a view towards a new Abel-Jacobi map.

The idea for the construction is similar to the one for generalized differential cohomology theories in [14]. Let X be a complex manifold. Another way to look at Deligne cohomology is to use the fact that the complex $\mathbb{Z}_{\mathcal{D}}(p)$ is quasi-isomorphic to the homotopy pullback of the diagram of sheaves of complexes

$$\begin{array}{ccc} & & \mathbb{Z} \\ & & \downarrow \\ \Omega_X^{*\geq p} & \longrightarrow & \Omega_X^* \end{array}$$

where Ω_X^* denotes the complex of holomorphic forms on X and $\Omega_X^{*\geq p}$ denotes the subcomplex of forms of degree at least p . We would like to replace the complex \mathbb{Z} which contributes the singular cohomology of X with a spectrum representing

a more general cohomology theory. In particular, we would like to use the Thom spectrum MU of complex bordism.

In order to make this work, we have to throw the latter chain complexes and MU in a common category. We do this by adding a simplicial direction and considering simplicial sets rather than chain complexes. Moreover, in order to have a good homotopy theory for complex manifolds as part of the team we consider presheaves of spectra on the site of complex manifolds with open coverings. In order to illustrate the basic pattern for the construction, we will start in the next section with a reformulation of Deligne cohomology in terms of simplicial presheaves before we move on to the generalizations.

The technical details for the general case are a bit more involved. In particular, to obtain a nice Eilenberg-MacLane spectrum functor from presheaves of differential graded algebras to presheaves of symmetric spectra which preserves the product structure requires more machinery on the pitch than you would like. The construction is based on the idea of Brown in [3] to use sheaves of spectra to define generalized sheaf cohomology.

Once our roster is complete, we can construct for every integer p and any symmetric spectrum E such that $\pi_j E \otimes \mathbb{C}$ vanishes if j is odd, a new presheaf of symmetric spectra $E_{\mathcal{D}}(p)$ as a suitable homotopy pullback and define the Hodge filtered complex bordism groups $E_{\mathcal{D}}^n(p)(X)$ of X as the homotopy groups of $E_{\mathcal{D}}(p)$.

Given a map of symmetric spectra $E \rightarrow H\mathbb{Z}$ from E to the integral Eilenberg-MacLane spectrum, we obtain a natural homomorphism

$$(2) \quad E_{\mathcal{D}}^n(p)(X) \rightarrow H_{\mathcal{D}}^n(X; \mathbb{Z}(p))$$

between Hodge filtered E -cohomology and Deligne cohomology groups of any complex manifold X . For $E = MU$, we will show that Hodge filtered complex bordism has a multiplicative structure and that map (2) respects the ring structures on both sides.

One motivation for this project is the result that for a compact complex Kähler manifold X the diagonal Hodge filtered cohomology groups $E_{\mathcal{D}}^{2p}(p)(X)$ sit in an exact sequence

$$(3) \quad 0 \rightarrow J_E^{2p-1}(X) \rightarrow E_{\mathcal{D}}^{2p}(p)(X) \rightarrow \mathrm{Hdg}_E^{2p}(X) \rightarrow 0.$$

The group on the right is the subset of elements in $E^{2p}(X)$ that map to Hodge classes in cohomology. The left hand group is an analogue of the intermediate Jacobian and carries the structure of a complex torus. Moreover, a map $E \rightarrow H\mathbb{Z}$ induces a natural map of short exact sequences from (1) to (3).

We have written the first five sections of this paper in a way that applies to complex manifolds. In the final section we specialize to the case of smooth algebraic varieties and modify our construction in order to take mixed Hodge structures into account. Following Deligne and Beilinson, we replace the sheaves of holomorphic forms occurring in the definition of $E_{\mathcal{D}}(p)$ with the sheaves of holomorphic forms having logarithmic singularities to form $E_{\log}(p)$. These "logarithmic" generalized cohomology groups are finitely generated if the groups $E_*(\mathrm{pt})$ are, and are \mathbb{A}^1 -invariant.

In the case that E is the topological Thom spectrum MU , we obtain a projective bundle formula and transfer maps for projective morphisms. This transfer structure provides a new cycle map that associates to a smooth irreducible subvariety Z of codimension p of a smooth projective complex variety X an element

in $MU_{\mathcal{D}}^{2p}(p)(X)$. This also induces a generalized Abel-Jacobi homomorphism on smooth cycles that map to zero in $\mathrm{Hdg}_{MU}^{2p}(X)$, a subgroup of the cycles homologically equivalent to zero. More generally, for every smooth quasi-projective complex variety X , we obtain a natural homomorphism from algebraic cobordism of Levine and Morel [20] to Hodge filtered complex bordism

$$\Omega^*(X) \rightarrow MU_{\log}^{2*}(X).$$

The sequence (3) indicates that there are a lot of torsion elements in $MU_{\log}^{2p}(p)(X)$ that are not seen by $MU^{2p}(X)$. Recalling that Totaro's factorization is only interesting in the presence of torsion elements this makes us optimistic that the Hodge filtered complex bordism can reveal new phenomena for the classical cycle map and the Abel-Jacobi-homomorphism. The study of the properties of the new cycle map is an ongoing project of the authors.

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2. DELIGNE COHOMOLOGY IN TERMS OF SIMPLICIAL PRESHEAVES

We would like to modify the definition of Deligne cohomology groups in a way that allows a generalization. The new theory should fit in an exact sequence similar to (1) for the role of singular cohomology is replaced by a generalized cohomology theory.

The starting observation is that we can consider the Deligne complex $\mathbb{Z}_{\mathcal{D}}(p)$ also as a homotopy pullback of complexes. Up to quasi-isomorphism $\mathbb{Z}_{\mathcal{D}}(p)$ fits into the homotopy pullback square of complexes

$$(4) \quad \begin{array}{ccc} \mathbb{Z}_{\mathcal{D}}(p) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \Omega_X^{*\geq p} & \longrightarrow & \Omega_X^* \end{array}$$

where Ω_X^* denotes the full complex of holomorphic differential forms and $\Omega_X^{*\geq p}$ is the truncated subcomplex of Ω_X^* of forms of degree at least p . The right hand vertical map in (4) is given by $(2\pi i)^p$ times the canonical inclusion $\mathbb{Z} \hookrightarrow \mathbb{C}$. Taking the homotopy pullback along the lower horizontal map corresponds to cutting the complex $0 \rightarrow \mathbb{Z} \rightarrow \Omega_X^*$ at degree p .

Now we would like to substitute the sheaf of chain complexes \mathbb{Z} by a different player. In order to set the stage for this replacement we consider diagram (4) in the world of simplicial presheaves on complex manifolds.

2.1. Simplicial presheaves. Let $\mathbf{Man}_{\mathbb{C}}$ be the category of complex manifolds. We consider it as a site with the Grothendieck topology defined by open coverings, or equivalently local homeomorphisms which are holomorphic maps.

Let \mathbf{Pre} be the category of presheaves of sets on the site $\mathbf{Man}_{\mathbb{C}}$. We denote by \mathbf{sPre} the category of simplicial objects in \mathbf{Pre} , or equivalently the category of presheaves of simplicial sets on $\mathbf{Man}_{\mathbb{C}}$. Sending an object U of $\mathbf{Man}_{\mathbb{C}}$ to the presheaf it represents defines a fully faithful embedding $\mathbf{Man}_{\mathbb{C}} \hookrightarrow \mathbf{Pre}$. Since any presheaf defines an object in \mathbf{sPre} of simplicial dimension zero, we can also embed $\mathbf{Man}_{\mathbb{C}}$ into \mathbf{sPre} . On the other hand every simplicial set defines a simplicial

presheaf. For example, the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$ can be considered as a simplicial presheaf by sending any connected object U to S^1 .

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of simplicial presheaves. Then f is called

- a (local) weak equivalence if it induces stalkwise a weak equivalence of simplicial sets at every point of $\mathbf{Man}_{\mathbb{C}}$;
- an injective cofibration if it induces a monomorphism of simplicial sets $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ for every object U of $\mathbf{Man}_{\mathbb{C}}$;
- a global fibration if it has the right lifting property with respect to any injective cofibration which is also a weak equivalence.

These classes of morphisms define a closed proper cellular simplicial model structure on \mathbf{sPre} (see [17, Theorem 2.3]). We denote its homotopy category by $\text{hos}\mathbf{Pre}$.

There are several other different interesting model structures on \mathbf{sPre} . In the local projective model structure the weak equivalences are again the local, i.e. stalkwise, weak equivalences. A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of simplicial presheaves is called

- a projective cofibration if it has the left lifting property with respect to maps which induce a trivial cofibration of simplicial sets $\mathcal{X}(U) \rightarrow \mathcal{Y}(U)$ for every object U ;
- a local projective fibration if it has the right lifting property with respect to every projective cofibration which is also a local weak equivalence.

The classes of local weak equivalences, projective cofibrations and local projective fibrations provide \mathbf{sPre} with the structure of a proper cellular simplicial model category (see [2] and [7]). The following fact shows that both model structures on \mathbf{sPre} play a nice one-two with each other.

Theorem 2.1. ([2], [7], [8]) *The identity functor is a left Quillen equivalence from the local projective model structure to the local injective model structure on \mathbf{sPre} .*

An interesting feature of the local projective structure is that one can detect its fibrant objects by a nice criterion (see [7] and [8]). Let M be a complex manifold and let $U_{\bullet} \rightarrow M$ a hypercover of M . A simplicial presheaf \mathcal{X} is said to satisfy descent for the hypercover $U_{\bullet} \rightarrow M$ if the natural map

$$\mathcal{X}(M) \rightarrow \text{holim}_n \mathcal{X}(U_n)$$

is a weak equivalence of simplicial sets. A simplicial presheaf \mathcal{X} is fibrant in the local projective model structure if it $\mathcal{X}(M)$ is a Kan complex for every object M of $\mathbf{Man}_{\mathbb{C}}$ and if it satisfies descent for all hypercovers. We will make use of this criterion when we show that the singular functor is a fibrant replacement in \mathbf{sPre} .

Finally, there is a third class of fibrations given by a local condition that is in fact easier to check and therefore convenient for applications. A map f of simplicial presheaves is called a local fibration (resp. local trivial fibration) if the map of stalks f_x is a Kan fibration (resp. Kan fibration and weak equivalence) of simplicial sets for every point x of $\mathbf{Man}_{\mathbb{C}}$. A simplicial presheaf \mathcal{X} is called locally fibrant if the unique map from \mathcal{X} to the final object is a local fibration.

For locally fibrant simplicial presheaves \mathcal{X} and \mathcal{Y} , let $\pi(\mathcal{X}, \mathcal{Y})$ be the quotient of $\text{Hom}_{\mathbf{sPre}}(\mathcal{X}, \mathcal{Y})$ with respect to the smallest equivalence relation generated by simplicial homotopies. The set $\pi(\mathcal{X}, \mathcal{Y})$ is called the set of simplicial homotopy classes of morphisms from \mathcal{X} to \mathcal{Y} . For a simplicial presheaf \mathcal{X} , denote by $\pi\text{Triv}/\mathcal{X}$ the category whose objects are the trivial local fibrations to \mathcal{X} and whose morphisms

are simplicial homotopy classes of morphisms which fit in the obvious commutative triangle over \mathcal{X} . A crucial fact about the category $\pi Triv$ is that it approximates the homotopy category \mathbf{hosPre} in the following sense.

Proposition 2.2. ([3, Proof of Theorem 2], [21, Proposition 2.1.13] and [17, p. 55]) *For any simplicial presheaves \mathcal{X} and \mathcal{Y} with \mathcal{Y} locally fibrant, the canonical map*

$$p: \mathcal{X}' \rightarrow \mathcal{X} \in \pi Triv / \mathcal{X} \quad \text{colim} \quad \pi(\mathcal{X}', \mathcal{Y}) \rightarrow \text{Hom}_{\mathbf{hosPre}}(\mathcal{X}, \mathcal{Y})$$

is a bijection.

We would like to be able to recover in \mathbf{hosPre} homotopy classes of maps between topological spaces. But the full embedding that sends an object of $\mathbf{Man}_{\mathbb{C}}$ to the presheaf in simplicial dimension zero that it represents is very rigid. For we have an isomorphism

$$\text{Hom}_{\mathbf{Man}_{\mathbb{C}}}(X, Y) \cong \text{Hom}_{\mathbf{hosPre}}(X, Y),$$

i.e. the functor $\mathbf{Man}_{\mathbb{C}} \rightarrow \mathbf{hosPre}$ is still a full embedding. The simplicial direction is crucial and there are at least two ways to use it. The first one is the singular functor of Suslin and Voevodsky [29] in the topological context.

Let Δ^n be the standard topological n -simplex

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_j \leq 1, \sum t_j = 1\}.$$

For topological spaces Y and Z , the singular function complex $\text{Sing}_*(Z, Y)$ is the simplicial set whose n -simplices are continuous maps

$$f : Z \times \Delta^n \rightarrow Y.$$

We denote the simplicial presheaf

$$U \mapsto \text{Sing}_*(Z, Y) =: \text{Sing}_*Y(Z)$$

by Sing_*Y . Note that Sing_*Y is a simplicial presheaf on our site. The singular simplicial set of a topological space Y is denoted by $\mathcal{S}(Y) \in \mathbf{sS}$. The geometric realization of a simplicial set K is denoted by $|K|$.

For topological spaces Y and Z , we denote by Y^Z the topological space of continuous maps $Z \rightarrow Y$ with the compact-open topology. Then the adjunction of taking products and mapping spaces yields a canonical isomorphism of simplicial sets

$$\text{Sing}_*(Z, Y) = \mathcal{S}(Y^Z).$$

Lemma 2.3. *Let Y be a CW-complex. The simplicial presheaf Sing_*Y is a fibrant object in the local projective model structure on \mathbf{sPre} .*

Proof. Since the singular complex of any topological space is a Kan complex, Sing_*Y is objectwise fibrant. By [8, Corollary 7.1], it remains to show that Sing_*Y satisfies descent for all hypercovers. So let X be a complex manifold and $U_{\bullet} \rightarrow X$ an open hypercover. By [9, Theorem 1.3], the induced map $\text{hocolim}_n U_n \rightarrow X$ is a weak equivalence. Since Y is fibrant and \mathcal{S} preserves weak equivalences, the induced map

$$\text{Sing}_*Y(X) \rightarrow \text{Sing}_*Y(\text{hocolim}_n U_n) = \text{holim}_n \text{Sing}_*Y(U_n)$$

is a weak equivalence too. Hence Sing_*Y satisfies descent for any open hypercover and is fibrant. \square

Proposition 2.4. *Let K be a simplicial set. The natural map*

$$K \rightarrow \mathrm{Sing}_*|K|$$

is a weak equivalence of simplicial presheaves.

Proof. The stalks of the simplicial presheaf K are all canonically isomorphic to K itself. In order to compute the stalks of $\mathrm{Sing}_*|K|$, x be a point in a complex manifold X and let D_ϵ be a small open disk of radius ϵ containing x . Then we have a canonical map

$$\mathrm{colim}_{\epsilon \rightarrow 0} \mathrm{Hom}(\Delta^\bullet, |K|^{D_\epsilon}) \rightarrow \mathrm{Hom}(\Delta^\bullet, |K|).$$

Since X is a manifold and hence locally contractible, the left hand side is isomorphic to the stalk $(\mathrm{Sing}_*|K|)_x$ at x . The right hand side is just the singular complex of the topological space $|K|$. Hence we have a map

$$(\mathrm{Sing}_*|K|)_x \rightarrow \mathcal{S}(|K|).$$

This map is a weak equivalence. This implies that the map $K \rightarrow (\mathrm{Sing}_*|K|)_x$ is a weak equivalence. Hence $K \rightarrow \mathrm{Sing}_*|K|$ is a weak equivalence of simplicial presheaves. \square

Lemma 2.3 and Proposition 2.4 show that $\mathrm{Sing}_*|K|$ is a fibrant replacement of K in the local projective model structure on \mathbf{sPre} .

Lemma 2.5. *The functor $\mathrm{Sing}_* : \mathbf{sS} \rightarrow \mathbf{sPre}$, $K \mapsto \mathrm{Sing}_*|K|$, sends weak equivalences in \mathbf{sS} to local weak equivalences.*

Proof. This follows immediately from the calculation of stalks in the proof of Proposition 2.4. \square

Proposition 2.6. *Let X be a complex manifold and K be a simplicial set. There is a natural bijection*

$$\mathrm{Hom}_{\mathrm{hosPre}}(X, \mathrm{Sing}_*|K|) \cong \mathrm{Hom}_{\mathrm{hoTop}}(X, |K|).$$

Proof. There are different ways to see this. One would be to apply Verdier's hypercovering theorem, which is possible since $\mathrm{Sing}_*|K|$ is stalkwise fibrant. Another way is to use the freedom to pick one of the various underlying model structures for the homotopy category of \mathbf{sPre} . In the local projective model structure, X is a cofibrant object, since it is representable. By Lemma 2.3, $\mathrm{Sing}_*|K|$ is fibrant in the local projective structure. Then we get the following sequence of natural bijections

$$\mathrm{Hom}_{\mathrm{hosPre}}(X, \mathrm{Sing}_*|K|) \cong \pi_0(\mathrm{Sing}_*|K|(X)) \cong \pi_0(\mathcal{S}(|K|^X)) \cong \mathrm{Hom}_{\mathrm{hoTop}}(X, |K|).$$

\square

As a consequence of Propositions 2.4 and 2.6, we obtain the following result about maps in the homotopy category to a simplicial presheaf given by a simplicial set.

Corollary 2.7. *Let X be a complex manifold and K be a simplicial set. There is a natural bijection*

$$\mathrm{Hom}_{\mathrm{hosPre}}(X, K) \cong \mathrm{Hom}_{\mathrm{hoTop}}(X, |K|).$$

For given n and an abelian group A , let $K(A, n)$ be an Eilenberg-MacLane space in \mathbf{sS} . For a topological space X , we denote by $H_{\text{sing}}^n(X; A)$ the n th singular cohomology with coefficients in A . It is isomorphic to the group of maps $X \rightarrow |K(A, n)|$ in the homotopy category \mathbf{hoTop} . From [14, Corollary D.13], we can read off the homotopy groups of the global sections of $\text{Sing}_*|K(A, n)|$.

Lemma 2.8. *For any complex manifold X , the i th homotopy group of the simplicial set $\text{Sing}_*|K(A, n)|(X)$ is given by a natural isomorphism*

$$\pi_i(\text{Sing}_*|K(A, n)|(X)) \cong H_{\text{sing}}^{n-i}(X; A).$$

Moreover, Corollary 2.7 shows that $K(A, n)$ represents singular cohomology for complex manifolds also in \mathbf{hosPre} .

Proposition 2.9. *For an abelian group A and a complex manifold X , there is a natural isomorphism*

$$\text{Hom}_{\mathbf{hosPre}}(X, |K(A, n)|) \cong H_{\text{sing}}^n(X; A).$$

Proof. By Propositions 2.4 and 2.6, the set $\text{Hom}_{\mathbf{hosPre}}(X, K(A, n))$ is in bijection with $\text{Hom}_{\mathbf{hoTop}}(X, |K(A, n)|) \cong H_{\text{sing}}^n(X; A)$. \square

Remark 2.10. For an abelian group A , we also denote by A the constant presheaf with value A . We denote the image of the complex with a single nontrivial sheaf A placed in degree n under the Dold-Kan correspondence by $K(A, n)$. This is just the constant simplicial presheaf with value the simplicial Eilenberg-MacLane space $K(A, n)$.

2.2. Hypercohomology. For a chain complex of presheaves of abelian groups \mathcal{C}_* on $\mathbf{Man}_{\mathbb{C}}$, we denote by $H_i(\mathcal{C}_*)$ the presheaf $U \mapsto H_i(\mathcal{C}_*(U))$. For a cochain complex \mathcal{C}^* we will denote by \mathcal{C}_* its associated chain complex given by $\mathcal{C}_n := \mathcal{C}^{-n}$. For any given n , we denote by $\mathcal{C}^*[n]$ the cochain complex given in degree q by $\mathcal{C}^q[n] := \mathcal{C}^{q-n}$.

Applying the normalized chain complex functor pointwise we obtain a functor $\mathcal{G} \mapsto N(\mathcal{G})$ from simplicial sheaves of abelian groups to chain complexes of sheaves of abelian groups. Then we have $\pi_i(\mathcal{G}) \cong H_i(N(\mathcal{G}))$. Moreover, the functor has a right adjoint Γ again obtained by applying the corresponding functor for chain complexes pointwise. The following result is the analog of the Dold-Kan correspondence for simplicial sheaves.

Proposition 2.11. ([3] and [21, Proposition 2.1.24]) *The pair (N, Γ) is a pair of mutually inverse equivalences between the category of complexes of sheaves of abelian groups \mathcal{C}_* with $\mathcal{C}_i = 0$ for $i < 0$ and the category of simplicial sheaves of abelian groups.*

Let \mathcal{F} be a sheaf of abelian groups. We denote the image of the complex with a single nontrivial sheaf \mathcal{F} placed in degree n under the Dold-Kan correspondence by $K(\mathcal{F}, n)$. We denote the sheaf cohomology of X with coefficients in the sheaf \mathcal{F} by $H^n(X; \mathcal{F})$.

If \mathcal{C}^* is a cochain complex of sheaves of abelian groups on \mathbf{T} , the hypercohomology $\mathbb{H}^*(U, \mathcal{C}^*)$ of an object U of $\mathbf{Man}_{\mathbb{C}}$ with coefficients in \mathcal{C}^* is the graded group of morphisms $\text{Hom}(\mathbb{Z}_U, \mathcal{C}^*)$ in the derived category of cochain complexes of sheaves on $\mathbf{Man}_{\mathbb{C}}$.

The following result is a version of Verdier's hypercovering theorem due to Ken Brown in [3, Theorem 2]. We refer the reader also to [21, Proposition 2.1.25] and [19] for more details.

Proposition 2.12. ([3, Theorem 2]) *Let \mathcal{C}^* be a cochain complex of sheaves of abelian groups on $\mathbf{Man}_{\mathbb{C}}$. Then for any integer n and any object U of $\mathbf{Man}_{\mathbb{C}}$ one has a canonical isomorphism*

$$\mathbb{H}^n(U; \mathcal{C}^*) \cong \mathrm{Hom}_{\mathrm{hosPre}}(U, \Gamma(\mathcal{C}^*[n])).$$

In particular, if $\mathcal{C}_ = \mathcal{F}$ is a sheaf of abelian groups, we have*

$$H^n(U; \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{hosPre}}(U, K(\mathcal{F}, n)).$$

2.3. Deligne cohomology in terms of simplicial presheaves. For $p \geq 0$, let $K(\mathbb{Z}, n) \rightarrow K(\mathbb{C}, n)$ be a map of simplicial sets whose cohomology class corresponds to the $(2\pi i)^p$ -multiple of the canonical inclusion $\mathbb{Z} \subset \mathbb{C}$, under the isomorphism

$$\mathrm{Hom}_{\mathrm{hosS}}(K(\mathbb{Z}, n), K(\mathbb{C}, n)) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{C}).$$

This induces a morphism of simplicial presheaves

$$K(\mathbb{Z}, n) \rightarrow K(\mathbb{C}, n).$$

As mentioned in Remark 2.10, the constant simplicial presheaf $K(\mathbb{C}, n)$ is the image of the Dold-Kan correspondence of the complex \mathbb{C} given by the constant presheaf \mathbb{C} in degree n . The canonical inclusion into the complex of sheaves $\Omega^*[n]$ of holomorphic forms induces a map of simplicial presheaves

$$K(\mathbb{C}, n) \rightarrow \Gamma(\Omega^*[n]).$$

Combining these maps we obtain a morphism of simplicial presheaves

$$K(\mathbb{Z}, n) \rightarrow \Gamma(\Omega^*[n]).$$

We define $K(\mathbb{Z}, n)(p)$ to be the homotopy pullback of the diagram of simplicial presheaves

$$(5) \quad \begin{array}{ccc} K(\mathbb{Z}, n)(p) & \longrightarrow & K(\mathbb{Z}, n) \\ \downarrow & & \downarrow \\ \Gamma(\Omega^{*\geq p}[n]) & \longrightarrow & \Gamma(\Omega^*[n]). \end{array}$$

For a simplicial presheaf \mathcal{X} , we call the abelian groups

$$\mathrm{Hom}_{\mathrm{hosPre}}(\mathcal{X}, K(\mathbb{Z}, n)(p))$$

the Hodge filtered $K(\mathbb{Z}, n)$ -cohomology groups of \mathcal{X} .

Proposition 2.13. *Let X be a complex manifold. The Hodge filtered $K(\mathbb{Z}, n)$ -cohomology groups of X agree with Deligne cohomology groups of X , i.e. for every n and $p \geq 0$, the map induced by the adjointness property of the Dold-Kan correspondence induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{hosPre}}(X, K(\mathbb{Z}, n)(p)) \cong H_{\mathcal{D}}^n(X; \mathbb{Z}(p)).$$

This result follows from the fact that the Dold-Kan correspondence induces a map of long exact sequences in which we know that all the maps but the one in question are isomorphisms. We will discuss this in a more general context in the next section (see Proposition 4.5 below).

3. STABLE HOMOTOPY THEORY FOR COMPLEX MANIFOLDS

3.1. Presheaves of symmetric spectra. Although most of the theory can be formulated in terms of simplicial presheaves, we need a more general setting in order to obtain a product structure in the next section. Therefore, we consider presheaves of symmetric spectra.

Let \mathbf{T} be a Grothendieck site whose underlying category is small. Let \mathbf{sPre}_* be the category of pointed simplicial presheaves on \mathbf{T} . A symmetric sequence in \mathbf{sPre}_* is a sequence of pointed simplicial presheaves $\mathcal{X}_0, \mathcal{X}_1, \dots$ with an action by the n th symmetric group Σ_n on \mathcal{X}_n . For two symmetric sequences \mathcal{X} and \mathcal{Y} , the tensor product $\mathcal{X} \otimes \mathcal{Y}$ is defined as the symmetric sequence given in degree n by

$$(\mathcal{X} \otimes \mathcal{Y})_n = \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} (\mathcal{X}_p \wedge \mathcal{Y}_q).$$

Let S^n denote the n -fold smash product of the simplicial circle

$$S^1 = \Delta^1 / \partial \Delta[1]$$

with itself. Considering S^n as a constant simplicial presheaf, we obtain a symmetric sequence (S^0, S^1, S^2, \dots) where Σ_n acts on S^n by permutation of the factors. We denote this symmetric space by \mathbb{S} .

A presheaf of symmetric spectra \mathcal{E} is given by a sequence of pointed presheaves \mathcal{E}_n with an action Σ_n for $n \geq 0$ together with Σ_n -equivariant maps $S^1 \wedge \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$ such that the composite

$$S^p \wedge \mathcal{E}_n \rightarrow S^{p-1} \wedge \mathcal{E}_{n+1} \rightarrow \dots \rightarrow \mathcal{E}_{n+p}$$

is $\Sigma_p \times \Sigma_n$ -equivariant for all $n, p \geq 0$. A map $\mathcal{E} \rightarrow \mathcal{F}$ of presheaves of symmetric spectra is a collection of Σ_n -equivariant maps $\mathcal{E}_n \rightarrow \mathcal{F}_n$ compatible with the structure maps of \mathcal{E} and \mathcal{F} . Hence we could say that a presheaf of symmetric spectra is a symmetric spectrum object in the category \mathbf{sPre}_* . This justifies denoting the category of presheaves of symmetric spectra on \mathbf{T} by

$$\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\mathbf{T}) = \mathrm{Sp}^\Sigma(\mathbf{sPre}_*).$$

Another way to think of a presheaf of symmetric spectra \mathcal{E} is that \mathcal{E} is a symmetric space together with the structure of an \mathbb{S} -module given by a map

$$m : \mathbb{S} \otimes \mathcal{E} \rightarrow \mathcal{E}.$$

The important advantage of symmetric spectra compared to, say, presheaves of Bousfield-Friedlander spectra is that $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ has a monoidal structure defined as follows. For two presheaves of symmetric spectra \mathcal{E} and \mathcal{F} , the smash product $\mathcal{E} \wedge \mathcal{F}$ is defined as the coequalizer

$$\mathcal{E} \otimes \mathbb{S} \otimes \mathcal{F} \rightrightarrows \mathcal{E} \otimes \mathcal{F} \rightarrow \mathcal{E} \otimes_{\mathbb{S}} \mathcal{F} =: \mathcal{E} \wedge \mathcal{F}.$$

The two maps in the diagram are given by the module structure of \mathcal{F} and the twisted module structure map of \mathcal{E}

$$\mathcal{E} \otimes \mathbb{S} \xrightarrow{\tau} \mathbb{S} \otimes \mathcal{E} \rightarrow \mathcal{E}.$$

Example 3.1. As for symmetric spectra in [15], we have the following basic examples of presheaves of spectra.

(a) Let Sp^Σ be the category of symmetric spectra of simplicial sets defined in [15]. Every symmetric spectrum $E \in \mathrm{Sp}^\Sigma$ defines a presheaf of symmetric spectra

given by the constant presheaf with value E . In particular, the sequence of constant simplicial presheaves (S^0, S^1, S^2, \dots) with the obvious structure maps also defines a presheaf of symmetric spectra which we also denote by \mathbb{S} and call it the symmetric sphere spectrum.

(b) For a pointed simplicial presheaf \mathcal{X} , we denote by $\Sigma^\infty \mathcal{X}$ the presheaf of symmetric spectra given by the sequence of pointed simplicial presheaves $S^n \wedge \mathcal{X}$ with the natural isomorphisms $S^1 \wedge S^n \wedge \mathcal{X} \rightarrow S^{n+1} \wedge \mathcal{X}$ and the diagonal action of Σ_n on $S^n \wedge \mathcal{X}$ coming from the left permutation action on S^n and the trivial action on \mathcal{X} .

(c) In particular, if X an object of \mathbf{T} , we can associate to X a presheaf of symmetric spectra $\Sigma_+^\infty(X)$, where X denotes the simplicial presheaf represented by X and the $+$ -subscript means that we add a disjoint basepoint.

(d) Furthermore, if \mathcal{E} is a presheaf of symmetric spectra and n an integer, then we denote by $\Sigma^n \mathcal{E}$ the n th suspension of \mathcal{E} whose k th space is \mathcal{E}_{k+n} .

(e) Let \mathcal{X} and \mathcal{Y} be presheaves of symmetric spectra. There is a function spectrum $\mathcal{H}om_{\mathbb{S}}(\mathcal{X}, \mathcal{Y})$ in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ defined as the limit of the diagram in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$

$$\mathrm{Hom}_\Sigma(\mathcal{X}, \mathcal{Y}) \rightrightarrows \mathrm{Hom}_\Sigma(\mathbb{S} \otimes \mathcal{X}, \mathcal{Y})$$

where the two arrows given by m^* and m_* respectively for the \mathbb{S} -module structure map $\mathbb{S} \otimes \mathcal{X} \rightarrow \mathcal{X}$. The endofunctor $\mathcal{H}om_{\mathbb{S}}(\mathcal{Y}, -)$ of $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is right adjoint to the functor $- \wedge \mathcal{Y}$.

The category $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ has a stable model structure defined in two steps. We equip \mathbf{sPre}_* either with the local injective or with the local projective model structure. In either case we obtain the following intermediate structure.

Definition 3.2. A map $f : \mathcal{E} \rightarrow \mathcal{F}$ in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is called a projective weak equivalence (respectively fibration) if each $f_n : \mathcal{E}_n \rightarrow \mathcal{F}_n$ is a weak equivalence (respectively fibration) in \mathbf{sPre}_* . A map is called a projective cofibration if it has the left lifting property with respect to all maps that are projective weak equivalences and projective fibrations.

For either the local injective or the local projective model structure on \mathbf{sPre}_* , the following result is a consequence of Hovey's general result [16, Theorems 8.2 and 8.3].

Proposition 3.3. *The classes of projective weak equivalence, projective fibrations and projective cofibrations define a proper cellular model structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ such that $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is a Sp^Σ -model category.*

Definition 3.4. ([16, Definition 8.7]) Define the set S of maps in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ to be

$$\{F_{n+1}(C \wedge S^1) \xrightarrow{\zeta_n^C} F_n C\}$$

as C runs through the domains and codomains of the generating cofibrations of \mathbf{sPre}_* , and $n \geq 0$, where the map ζ_n^C is adjoint to the map

$$C \wedge S^1 \rightarrow \mathrm{Ev}_{n+1} F_n C = \Sigma_{n+1} \times (C \wedge S^1)$$

corresponding to the identity of Σ_{n+1} . The *stable local injective model structure* (respectively *stable local projective model structure*) on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is defined to be the left Bousfield localization with respect to S of the strict local injective (respectively local projective) model structure.

Since both the local projective and local injective model structures are proper, simplicial and cellular whose generating cofibrations and trivial cofibrations have cofibrant domains and codomains, we can deduce the following consequence of [16, Theorem 9.3], and Theorem 2.1.

Theorem 3.5. ([2], [7], [16], [18]) *The identity functor on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is a left Quillen equivalence from the stable local projective model structure to the stable local injective model structure. We denote the stable homotopy category obtained by localizing $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ at the stable local equivalences by $\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)$.*

Definition 3.6. (1) A presheaf of symmetric spectra $\mathcal{E} \in \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is called an Ω -spectrum if each \mathcal{E}_n is fibrant in \mathbf{sPre}_* and the adjoint structure maps $\mathcal{E}_n \rightarrow \mathcal{E}_{n+1}^{S^1}$ are weak equivalences in \mathbf{sPre}_* for all $n \geq 0$.

(2) A map in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is called an injective fibration if it has the right lifting property with respect to all maps that are level cofibrations and level weak equivalences.

(3) A spectrum \mathcal{E} in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is called an injective spectrum if the map $\mathcal{E} \rightarrow *$ is an injective fibration.

By [16, Theorem 8.8], the stably fibrant objects are the Ω -spectra. For two presheaves of symmetric spectra \mathcal{X} and \mathcal{Y} , let $\mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(\mathcal{X}, \mathcal{Y})$ denote the mapping space which is part of the simplicial structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. The following lemma shows that the identity functor on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is a Quillen equivalence from the stable local injective model structure to the stable model structure of [18].

Lemma 3.7. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. Then the following conditions are equivalent:*

- (1) *f is a stable equivalence.*
- (2) *f induces a weak equivalence of Kan complexes*

$$\mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(f, \mathcal{E}) : \mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(\mathcal{Y}, \mathcal{E}) \rightarrow \mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(\mathcal{X}, \mathcal{E})$$

for every injective Ω -spectrum \mathcal{E} in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_)$.*

- (3) *f induces a level equivalence*

$$\mathrm{Hom}_{\mathbb{S}}(f, \mathcal{E}) : \mathrm{Hom}_{\mathbb{S}}(\mathcal{Y}, \mathcal{E}) \rightarrow \mathrm{Hom}_{\mathbb{S}}(\mathcal{X}, \mathcal{E})$$

for every injective Ω -spectrum \mathcal{E} in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_)$.*

Proof. In order to prove the lemma we recall the injective model structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ of [18, Theorem 2]. A map in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ is called an injective cofibration (injective weak equivalence) if it is a levelwise cofibration (weak equivalence). The fibrant objects in the injective model structure are the injective Ω -spectra.

The identity functor provides a Quillen equivalence between the injective and projective model structures on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. In particular, every injective Ω -spectrum is a fibrant object in the projective model structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$ and every projective cofibrant object is also injective cofibrant. This has the following consequence.

Let Q_{proj} be a cofibrant replacement functor in the projective model structure. If f is a stable equivalence, then $\mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(Q_{\mathrm{proj}}f, \mathcal{E})$ is a weak equivalence of Kan complexes for every injective Ω -spectrum \mathcal{E} . Since the maps $\mathcal{X} \rightarrow Q_{\mathrm{proj}}\mathcal{X}$ and $\mathcal{Y} \rightarrow Q_{\mathrm{proj}}\mathcal{Y}$ are level equivalences, we obtain that $\mathrm{Map}_{\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)}(f, \mathcal{E})$ is a weak equivalence of Kan complexes for every injective Ω -spectrum \mathcal{E} .

Now let R_{inj} be a fibrant replacement functor in the injective model structure and assume that f induces a weak equivalence $\text{Map}_{\text{Sp}^\Sigma(\mathbf{sPre}_*)}(f, \mathcal{E})$ for every injective Ω -spectrum \mathcal{E} . This implies that $\text{Map}_{\text{Sp}^\Sigma(\mathbf{sPre}_*)}(Q_{\text{proj}}f, R_{\text{inj}}\mathcal{F})$ is a weak equivalence for every Ω -spectrum \mathcal{F} . Since the map $\mathcal{F} \rightarrow R_{\text{inj}}\mathcal{F}$ is a level equivalence, this shows that $\text{Map}_{\text{Sp}^\Sigma(\mathbf{sPre}_*)}(Q_{\text{proj}}f, \mathcal{F})$ is a weak equivalence for every Ω -spectrum \mathcal{F} , i.e. that f is a stable equivalence. This proves that (1) and (2) are equivalent.

The second and third conditions are equivalent, since

$$\text{Ev}_k \mathcal{H}om_{\mathbb{S}}(f, \mathcal{E}) = \text{Map}_{\text{Sp}^\Sigma(\mathbf{sPre}_*)}(f, \mathcal{H}om_{\mathbb{S}}(F_k S^0, \mathcal{E})).$$

□

Consider $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ with the stable local injective model structure. Let K be the class in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ of all maps $f \wedge \mathcal{X}$, where f is a stable trivial cofibration and \mathcal{X} is a presheaf of symmetric spectra. One can show exactly as in [15, §5.4], that that stable local injective model structure is monoidal and that each map in K -cof is a stable equivalence. This fact implies that the monoid axiom holds in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ and by [26, Theorem 4.1], we can deduce the following result on monoids in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$.

Theorem 3.8. *The forgetful functor creates a model structure on the category of monoids in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ for which a morphism is a weak equivalence (fibration) if and only if the underlying map of presheaves of symmetric spectra is a stable equivalence (stable fibration).*

Remark 3.9. The functor $\text{Sp}^\Sigma \rightarrow \text{Sp}^\Sigma(\mathbf{sPre}_*)$ sending a symmetric spectrum E to the constant presheaf of symmetric spectra with value E is a strong symmetric monoidal left Quillen functor. In particular, a commutative \mathbb{S} -algebra E in Sp^Σ is still a commutative \mathbb{S} -algebra in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ when we consider E as a constant presheaf.

3.2. Eilenberg-MacLane spectra. In the remainder of this section, we start to explore Brown's idea [3] to use sheaves of spectra to define generalized sheaf cohomology.

Let $\mathbf{PreDGA}_{\mathbb{C}}$ be the category of presheaves of differential graded \mathbb{C} -algebras on \mathbf{T} . By [28], for every differential graded algebra A_* , there is functorial construction of a symmetric Eilenberg-MacLane ring spectrum HA_* . (The reader may also want to consult [10, §2.6] for a short summary on the construction of the functor H .) Pointwise application yields a functor

$$H : \mathbf{PreDGA}_{\mathbb{C}} \rightarrow \text{Sp}^\Sigma(\mathbf{sPre}_*).$$

By its construction, the functor H sends the constant presheaf \mathbb{C} to the constant symmetric Eilenberg-MacLane spectrum HC . Moreover, since the tensor product in $\mathbf{PreDGA}_{\mathbb{C}}$ and the smash product in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$ are defined pointwise, the functor H is a lax monoidal functor and its image lies in fact in the subcategory of monoids over HC , or in other words HC -algebras in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$:

$$H : \mathbf{PreDGA}_{\mathbb{C}} \rightarrow HC - \mathbf{Alg}.$$

In particular, a morphism $\mathcal{C}^* \rightarrow \mathcal{D}^*$ of presheaves of differential graded \mathbb{C} -algebras induces a morphism of monoids $HC^* \rightarrow HD^*$ in $\text{Sp}^\Sigma(\mathbf{sPre}_*)$.

Our applications in the following sections require graded versions of the complex of holomorphic forms, since different coefficient rings will be substituted in the

game. The correct choice of compatible gradings and filtrations is an important point. This will become apparent in the proof of the main result in Theorem 4.13.

So we enlarge our roster by considering also cochains, cocycles and forms with values in an evenly graded complex vector space \mathcal{V} . The main example will be $\mathcal{V}_{2*} = \pi_{2*}MU \otimes \mathbb{C}$. We will use the convention to grade cochains and forms with values in \mathcal{V} in such a way that $C^i(X; \mathcal{V}_j)$ has total degree $(i - j)$. We will write

$$C^*(X; \mathcal{V}_{2*})^n = \bigoplus_j C^{n+2j}(X; \mathcal{V}_{2j}),$$

and

$$H^*(X; \mathcal{V}_{2*})^n = \bigoplus_j H^{n+2j}(X; \mathcal{V}_{2j}).$$

For the presheaf of holomorphic forms we will write

$$\Omega^*(\mathcal{V}_{2*}) = \bigoplus_j \Omega^*(\mathcal{V}_{2j})[-2j].$$

For a given integer p , we will denote

$$\Omega^{*\geq p}(\mathcal{V}_{2*}) := \bigoplus_j \Omega^{*\geq p+j}(\mathcal{V}_{2j})[-2j].$$

Note that by our grading convention, this defines in general a complex different from $\Omega^*(\mathcal{V}_{2*})$ even for negative p .

For hypercohomology groups we write

$$\mathbb{H}^*(X; \Omega^{*\geq p}(\mathcal{V}_{2*}))^n = \bigoplus_j \mathbb{H}^{n+2j}(X; \Omega^{*\geq p+j}(\mathcal{V}_{2j})[-2j]).$$

These gradings have corresponding counterparts on the level of Eilenberg-MacLane spectra. Let \mathcal{V}_{2*} be an evenly graded \mathbb{C} -algebra and $\mathcal{C}^* \in \mathbf{PreDGA}_{\mathbb{C}}$. We denote by $\mathcal{C}^*(\mathcal{V}_{2*})$ the presheaf of complexes given in degree n by

$$\mathcal{C}^*(\mathcal{V}_{2*})^n = \bigoplus_j \mathcal{C}^{n+2j} \otimes \mathcal{V}_{2j}.$$

The image of $\mathcal{C}^*(\mathcal{V}_{2*})$ under H in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$ has the form

$$H(\mathcal{C}^*(\mathcal{V}_{2*})) = \bigvee_j \Sigma^{2j} H(\mathcal{C}^* \otimes \mathcal{V}_{2j}).$$

In particular, if $\mathcal{C}^*(\mathcal{V}_{2*})$ is just the graded ring \mathcal{V}_{2*} considered as a chain complex concentrated in degree zero, then we have

$$H(\mathcal{V}_{2*}) = \bigvee_j \Sigma^{2j} H(\mathcal{V}_{2j}).$$

For example, let $\mathcal{C}^* = \Omega^*$ be the presheaf of holomorphic forms on $\mathbf{Man}_{\mathbb{C}}$. The inclusion of cochain complexes $\mathbb{C} \hookrightarrow \Omega^*$ induces a map of presheaves of symmetric spectra

$$H\mathbb{C} \rightarrow H\Omega^*.$$

Moreover, for the \mathbb{C} -algebra \mathcal{V}_{2*} , there is a canonical map of presheaves of symmetric spectra

$$H(\mathcal{V}_{2*}) \rightarrow H(\Omega^*(\mathcal{V}_{2*})).$$

Proposition 3.10. *Let X be an object of \mathbf{T} , $\mathcal{C}^* \in \mathbf{PreDGA}_{\mathbb{C}}$ and let \mathcal{V}_{2*} be an evenly graded commutative \mathbb{C} -algebra. Then we have a natural isomorphism of graded commutative rings*

$$\bigoplus_n \mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\Sigma_+^{\infty}(X), \Sigma^n H(\mathcal{C}^*(\mathcal{V}_{2*}))) \cong \bigoplus_n \mathbb{H}(X; \mathcal{C}^*(\mathcal{V}_{2*}))^n.$$

In particular, for $X \in \mathbf{Man}_{\mathbb{C}}$, there are natural isomorphisms of graded commutative rings

$$\begin{aligned} \bigoplus_n \mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\Sigma_+^{\infty}(X), \Sigma^n H(\Omega^*(\mathcal{V}_{2*}))) &\cong \bigoplus_n H(X; \mathcal{V}_{2*})^n \text{ and} \\ \bigoplus_n \mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\Sigma_+^{\infty}(X), \Sigma^n H(\Omega^{*\geq p}(\mathcal{V}_{2*}))) &\cong \bigoplus_n \mathbb{H}(X; \Omega^{*\geq p}(\mathcal{V}_{2*}))^n. \end{aligned}$$

Proof. Let $H_{\mathrm{nomo}} : \mathrm{Ch}_{\mathbb{Z}} \rightarrow \mathrm{Sp}^{\Sigma}$ be the non-monoidal Eilenberg-MacLane spectrum functor that sends a chain complex C to a symmetric spectrum which represents cohomology with coefficients in C . By [27, Proposition 5.1], the canonical map $H_{\mathrm{nomo}}C \rightarrow HC$ is an equivalence of underlying symmetric spectra. Hence, for every object $U \in \mathbf{T}$, the canonical map

$$H_{\mathrm{nomo}}(\mathcal{C}^*(\mathcal{V}_{2*}))(U) \rightarrow H(\mathcal{C}^*(\mathcal{V}_{2*}))(U)$$

is an equivalence of symmetric spectra. By abuse of notation, we denote the point-wise application of H_{nomo} again by H_{nomo} . Then the induced map of presheaves of symmetric spectra

$$(6) \quad H_{\mathrm{nomo}}(\mathcal{C}^*(\mathcal{V}_{2*})) \rightarrow H(\mathcal{C}^*(\mathcal{V}_{2*}))$$

is a stable equivalence in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$. Though the Eilenberg-MacLane spectrum $H_{\mathrm{nomo}}(\mathcal{C}^*(\mathcal{V}_{2*}))$ is not a monoid in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$, it is a ring in the stable homotopy category of presheaves of spectra. Moreover, equivalence (6) induces an isomorphism of rings in the stable homotopy category of presheaves of spectra between $H_{\mathrm{nomo}}(\mathcal{C}^*(\mathcal{V}_{2*}))$ and $H(\mathcal{C}^*(\mathcal{V}_{2*}))$. Hence it suffices to show that

$$\mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\Sigma^n \Sigma_+^{\infty}(X), H_{\mathrm{nomo}}(\mathcal{C}^*(\mathcal{V}_{2*}))) \cong \mathbb{H}^n(X; \mathcal{C}^*(\mathcal{V}_{2*}))$$

is an isomorphism.

Since Σ^{∞} is left Quillen adjoint to the evaluation functor at the 0th space, it suffices to show that we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{hosPre}_*}(\Sigma^n(X_+), \Gamma(\mathcal{C}^*(\mathcal{V}_{2j}))) \cong \mathbb{H}^{n-2j}(X; \mathcal{C}^*(\mathcal{V}_{2j})).$$

But, using our grading conventions, this is the content of Proposition 2.12. \square

3.3. The singular functor for symmetric spectra. In this subsection we work again on the site $\mathbf{T} = \mathbf{Man}_{\mathbb{C}}$. Our goal now is to show that a constant presheaf of symmetric spectra E represents E -cohomology also in $\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)$. There are different ways to show this. We use the singular functor as our favorite tool. provides a third kind of examples of objects in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$.

Let $\mathrm{Sp}_{\mathrm{Top}}^{\Sigma}$ be the category of symmetric spectra of topological spaces. Let X be a complex manifold and $F \in \mathrm{Sp}_{\mathrm{Top}}^{\Sigma}$. The levelwise defined spectrum F^X of functions from X to F is again a symmetric spectrum of topological spaces. Applying the

singular simplicial set functor levelwise yields then a symmetric spectrum of simplicial sets $\mathcal{S}(F^X)$ given in degree n by $\mathcal{S}(F_n^X)$. We denote the object in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$, defined by the functor

$$X \mapsto \mathcal{S}(F^X)$$

by $\mathrm{Sing}_* F$. Moreover, for $E \in \mathrm{Sp}^\Sigma$ let $|E|$ be the geometric realization of E in $\mathrm{Sp}_{\mathrm{Top}}^\Sigma$. Then Sing_* also defines a functor

$$\mathrm{Sing}_* : \mathrm{Sp}^\Sigma \rightarrow \mathrm{Sp}^\Sigma(\mathbf{sPre}_*), \quad E \mapsto \mathrm{Sing}_* |E|.$$

Proposition 3.11. *Let $E \in \mathrm{Sp}^\Sigma$ be a symmetric spectrum. We also denote by E the constant presheaf with value E . The natural map*

$$E \rightarrow \mathrm{Sing}_* |E|$$

is a levelwise equivalence and hence a stable equivalence in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_)$.*

Proof. By Proposition 2.4, each map $E_n \rightarrow \mathrm{Sing}_* |E|_n \cong (\mathrm{Sing}_* |E|)_n$ is a weak equivalence in \mathbf{sPre}_* . Hence $E \rightarrow \mathrm{Sing}_* |E|$ is a levelwise equivalence in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. \square

Lemma 3.12. *For any spectrum $E \in \mathrm{Sp}^\Sigma$, the presheaf of symmetric spectra $\mathrm{Sing}_* |E|$ is fibrant in the stable local projective model structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$.*

Proof. By Lemma 2.3, each space $\mathrm{Sing}_* |E|_n$ is fibrant in the local projective model structure on \mathbf{sPre}_* . Moreover, for any complex manifold X , the induced map

$$\mathcal{S}(|E_n|^X) \rightarrow \mathcal{S}((\Omega |E_{n+1}|)^X) \cong \Omega(\mathcal{S}(|E_{n+1}|^X))$$

is again a weak equivalence of simplicial sets. Hence $\mathrm{Sing}_* |E|_n \rightarrow \Omega \mathrm{Sing}_* |E|_{n+1}$ is a weak equivalence in \mathbf{sPre}_* . \square

Proposition 3.13. *Let E be a symmetric spectrum and let X be a complex manifold. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)}(\Sigma_+^\infty(X), \mathrm{Sing}_* |E|) \cong \mathrm{Hom}_{\mathrm{hoSp}_{\mathrm{Top}}^\Sigma}(\Sigma_+^\infty(X), |E|).$$

Proof. By Lemma 3.12, $\mathrm{Sing}_* |E|$ is fibrant in the stable local projective model structure on $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$. Since $\Sigma_+^\infty(X)$ is cofibrant, there is a sequence of natural bijections

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)}(\Sigma_+^\infty(X), \mathrm{Sing}_* |E|) \cong \pi_0(\mathcal{S}(|E|^X)) \cong \mathrm{Hom}_{\mathrm{hoSp}_{\mathrm{Top}}^\Sigma}(\Sigma_+^\infty(X), |E|).$$

\square

Propositions 3.11 and 3.13 show that the constant presheaf of symmetric given by a symmetric spectrum E represents E -cohomology for manifolds also in $\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)$.

Corollary 3.14. *Let E be a symmetric spectrum and let X be a complex manifold. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)}(\Sigma_+^\infty(X), E) \cong \mathrm{Hom}_{\mathrm{hoSp}_{\mathrm{Top}}^\Sigma}(\Sigma_+^\infty(X), |E|).$$

4. HODGE FILTERED COHOMOLOGY GROUPS

4.1. The definition. In this section we work on the site $\mathbf{T} = \mathbf{Man}_{\mathbb{C}}$ of complex manifolds with the analytic topology.

Definition 4.1. A *rationally even spectrum* is a symmetric spectrum E with the property that $\pi_j E \otimes \mathbb{Q} = 0$ if j is odd.

Let $E \in \mathrm{Sp}^{\Sigma}$ be a rationally even spectrum such that $\pi_j E \otimes \mathbb{C}$ vanishes if j is odd. Let

$$\iota : E \rightarrow E \wedge H\mathbb{C}$$

be a map in Sp^{Σ} which induces for every n the map

$$\pi_{2n}(E) \xrightarrow{(2\pi i)^n} \pi_{2n}(MU_{\mathbb{C}})$$

defined by multiplication by $(2\pi i)^n$ on homotopy groups.

Let p be an integer. Multiplication by $(2\pi i)^p$ on homotopy groups determines a map

$$E \xrightarrow{(2\pi i)^p \iota} E \wedge H\mathbb{C}.$$

Let

$$E \wedge H\mathbb{C} \rightarrow H(\pi_{2*} E \otimes \mathbb{C})$$

be a map in Sp^{Σ} that induces the isomorphism

$$\pi_{2*}(E \wedge H\mathbb{C}) \cong \pi_{2*} E \otimes \mathbb{C}.$$

Let

$$(7) \quad \tau^E : E \rightarrow H(\pi_{2*} E \otimes \mathbb{C})$$

be the composition considered as a map in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$.

Composition with the canonical map

$$H(\pi_{2*} E \otimes \mathbb{C}) \rightarrow H(\Omega^*(\pi_{2*} E \otimes \mathbb{C}))$$

yields a map of presheaves of symmetric spectra on $\mathbf{Man}_{\mathbb{C}}$

$$(8) \quad E \rightarrow H(\Omega^*(\pi_{2*} E \otimes \mathbb{C})).$$

Given an integer p , we define the presheaf of symmetric spectra $E_{\mathcal{D}}(p)$ to be the homotopy pullback in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$ of the diagram

$$(9) \quad \begin{array}{ccc} E_{\mathcal{D}}(p) & \xrightarrow{\quad} & E \\ \downarrow & & \downarrow \\ H(\Omega^{*\geq p}(\pi_{2*} E \otimes \mathbb{C})) & \longrightarrow & H(\Omega^*(\pi_{2*} E \otimes \mathbb{C})). \end{array}$$

Definition 4.2. Let n and p be integers, \mathcal{X} be a presheaf of symmetric spectra on $\mathbf{Man}_{\mathbb{C}}$, and E be a rationally even spectrum. The *Hodge filtered E -cohomology groups* $E_{\mathcal{D}}^n(p)(\mathcal{X})$ are defined as

$$E_{\mathcal{D}}^n(p)(\mathcal{X}) := \mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\mathcal{X}, \Sigma^n E_{\mathcal{D}}(p)).$$

Remark 4.3. It follows from the definition of $E_{\mathcal{D}}(p)$ and the splitting of ordinary cohomology theories that rationally Hodge filtered complex cohomology splits in the same way into a sum of rational Deligne cohomology groups as rational E -cohomology splits into a sum of ordinary cohomology. More precisely, the definitions are chosen such that there is an isomorphism

$$(10) \quad (E \wedge H\mathbb{Q})_{\mathcal{D}}^n(p)X \cong \bigoplus_j H_{\mathcal{D}}^{n+2j}(X; \mathbb{Q}(p+j)) \otimes \pi_{2j}E.$$

Remark 4.4. For $p = 0$, the Deligne complex $\mathbb{Z}_{\mathcal{D}}(0)$ is canonically isomorphic to \mathbb{Z} and the cohomology groups $H_{\mathcal{D}}^n(X; \mathbb{Z}(0))$ are isomorphic to integral cohomology $H^n(X; \mathbb{Z})$. But for generalized Hodge filtered E -cohomology, it is in general not true that we recover ordinary E -cohomology groups if $p = 0$. The bottom row in the diagram

$$(11) \quad \begin{array}{ccc} E_{\mathcal{D}}(0) & \longrightarrow & E \\ \downarrow & & \downarrow \\ H(\Omega^{*\geq 0}(\pi_{2*}MU \otimes \mathbb{C})) & \longrightarrow & H(\Omega^*(\pi_{2*}E \otimes \mathbb{C})) \end{array}$$

is not an equivalence. By definition of

$$H(\Omega^{*\geq 0}(\pi_{2*}E \otimes \mathbb{C})) = \bigvee_j \Sigma^{2j} H(\Omega^{*\geq j}(\pi_{2j}E \otimes \mathbb{C}))$$

the step of the Hodge filtration that we divide out depends on the grading of $\pi_{2*}E \otimes \mathbb{C}$. In particular, the top row in diagram (11) defining $E_{\mathcal{D}}(0)$ is not an equivalence.

4.2. Functoriality and Mayer-Vietoris sequence. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a map of presheaves of symmetric spectra $f : \mathcal{X} \rightarrow \mathcal{Y}$. Then it follows from definition that there is an induced pullback map

$$f^* : E_{\mathcal{D}}^n(p)(\mathcal{Y}) \rightarrow E_{\mathcal{D}}^n(p)(\mathcal{X}).$$

In particular, if $\rho : \mathcal{U} \rightarrow X$ is a hypercovering of a complex manifold X , then ρ induces an isomorphism

$$(12) \quad \rho^* : E_{\mathcal{D}}^n(p)(X) \xrightarrow{\cong} E_{\mathcal{D}}^n(p)(\mathcal{U}).$$

Furthermore, it is a general fact that when \mathcal{Z} is the homotopy pullback of a diagram in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$

$$\begin{array}{ccc} & \mathcal{U} & \\ & \downarrow & \\ \mathcal{V} & \longrightarrow & \mathcal{W}, \end{array}$$

there is an induced long exact sequence

$$\dots \rightarrow [\mathcal{X}, \Omega\mathcal{U}] \oplus [\mathcal{X}, \Omega\mathcal{V}] \rightarrow [\mathcal{X}, \Omega\mathcal{W}] \rightarrow [\mathcal{X}, \mathcal{Z}] \rightarrow [\mathcal{X}, \mathcal{U}] \oplus [\mathcal{X}, \mathcal{V}] \rightarrow [\mathcal{X}, \mathcal{W}].$$

Since $E(p)$ is defined as the homotopy pullback of (9), the Hodge filtered cohomology groups sit in a long exact sequence. For a complex manifold this sequence has the following form.

Proposition 4.5. *For any complex manifold X , Hodge filtered E -cohomology groups sit in a long exact sequence*

$$\begin{aligned} \dots \rightarrow H^*(X; \pi_{2*}E \otimes \mathbb{C})^{n-1} &\rightarrow E_{\mathcal{D}}^n(p)(X) \rightarrow \\ \rightarrow E^n(X) \oplus \mathbb{H}^*(X; \Omega^{*\geq p}(\pi_{2*}E \otimes \mathbb{C}))^n &\rightarrow H^*(X; \pi_{2*}E \otimes \mathbb{C})^n \rightarrow \dots \end{aligned}$$

Proof. By the above fact, it suffices to determine the groups in the long exact sequence resulting from the definition of $E_{\mathcal{D}}(p)$ as the homotopy pullback of (9). This can be done using Proposition 3.13 for $E^n(X)$ and using Proposition 3.10 for the hypercohomology groups. \square

Remark 4.6. If X is a compact complex Kähler manifold, then the hypercohomology group of $\Omega^{*\geq p}(\pi_{2*}E \otimes \mathbb{C})$ in the long exact sequence of Proposition 4.5 detects the Hodge filtration, i.e. there is an isomorphism

$$\mathbb{H}^*(X; \Omega^{*\geq p}(\pi_{2*}E \otimes \mathbb{C}))^n \cong \bigoplus_j F^{p+j} H^{n+2j}(X; \pi_{2j}E \otimes \mathbb{C}) =: F^{p+*} H^*(X; \pi_{2*}E \otimes \mathbb{C})^n.$$

But for an arbitrary complex manifold X , the hypercohomology of $\Omega^{*\geq p}(\pi_{2*}E \otimes \mathbb{C})$ is a much less well-behaved group. We will see below how this defect can be fixed for smooth complex algebraic varieties.

For a cofibration of presheaves of symmetric spectra $i : \mathcal{Y} \rightarrow \mathcal{X}$, for example the map induced by a monomorphism of simplicial presheaves, let \mathcal{X}/\mathcal{Y} be the quotient in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$. We define relative Hodge filtered E -cohomology groups to be

$$(13) \quad E_{\mathcal{D}}^n(p)(\mathcal{X}, \mathcal{Y}) := \mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\mathcal{X}/\mathcal{Y}, \Sigma^n E_{\mathcal{D}}(p)).$$

Just as in the topology, one proves the following result.

Proposition 4.7. *These relative groups sit in a long exact sequence*

$$(14) \quad \dots \rightarrow E_{\mathcal{D}}^{n-1}(p)(\mathcal{X}, \mathcal{Y}) \rightarrow E_{\mathcal{D}}^n(p)(\mathcal{X}) \rightarrow E_{\mathcal{D}}^n(p)(\mathcal{Y}) \rightarrow E_{\mathcal{D}}^n(p)(\mathcal{X}, \mathcal{Y}) \rightarrow \dots$$

induced by the cofiber sequence $\mathcal{Y} \rightarrow \mathcal{X} \rightarrow \mathcal{Y}/\mathcal{X}$.

Moreover, there is a Mayer-Vietoris type long exact sequence for Hodge filtered cohomology groups.

Proposition 4.8. *Let X be a complex manifold and U and V be two open submanifolds such that $U \cup V = X$. Then there is a long exact Mayer-Vietoris sequence*

$$\dots \rightarrow E_{\mathcal{D}}^n(p)(X) \rightarrow E_{\mathcal{D}}^n(p)(U) \oplus E_{\mathcal{D}}^n(p)(V) \rightarrow E_{\mathcal{D}}^n(p)(U \cap V) \rightarrow E_{\mathcal{D}}^{n+1}(p)(X) \rightarrow \dots$$

Proof. The long exact sequence is induced by applying the functor

$$\mathrm{Hom}_{\mathrm{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\Sigma_+^{\infty}(-), \Sigma^n E_{\mathcal{D}}(p))$$

to the pushout diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

of simplicial presheaves corresponding to the covering of X by U and V . \square

Finally, let $g : E \rightarrow F$ be a morphism of rationally even spectra and $\tau^F : F \rightarrow H(\pi_{2*}F \otimes \mathbb{C})$ be a map with the properties described at the beginning of the previous section (for τ^E). The pullback of τ^F along g defines a map $g^*\tau^F : E \rightarrow H(\pi_{2*}E \otimes \mathbb{C})$ and f induces a commutative diagram of presheaves of symmetric spectra

$$\begin{array}{ccc} E & \xrightarrow{\quad} & F \\ g^*\tau^F \downarrow & & \downarrow \tau^F \\ H(\pi_{2*}E \otimes \mathbb{C}) & \xrightarrow{\quad} & H(\pi_{2*}F \otimes \mathbb{C}) \\ \downarrow & & \downarrow \\ H(\Omega^*(\pi_{2*}E \otimes \mathbb{C})) & \xrightarrow{\quad} & H(\Omega^*(\pi_{2*}F \otimes \mathbb{C})). \end{array}$$

Together with the induced map on filtered forms we obtain a commutative diagram

$$(15) \quad \begin{array}{ccccc} E & \xrightarrow{\quad} & H(\Omega^*(\pi_{2*}E \otimes \mathbb{C})) & \longleftarrow & H(\Omega^{\geq p}(\pi_{2*}E \otimes \mathbb{C})) \\ \downarrow & & \downarrow & & \downarrow \\ F & \xrightarrow{\quad} & H(\Omega^*(\pi_{2*}F \otimes \mathbb{C})) & \longleftarrow & H(\Omega^{\geq p}(\pi_{2*}F \otimes \mathbb{C})). \end{array}$$

This diagram induces a map of homotopy pullbacks

$$g(p) : E_{\mathcal{D}}(p) \rightarrow F_{\mathcal{D}}(p)$$

and hence map of Hodge filtered cohomology groups

$$g_{\mathcal{D}}(p) : E_{\mathcal{D}}^n(p)(\mathcal{X}) \rightarrow F_{\mathcal{D}}^n(p)(\mathcal{X})$$

for every presheaf of symmetric spectra \mathcal{X} .

Proposition 4.9. *Let $g : E \rightarrow F$ be a stable equivalence of spectra and let $\tau^F : F \rightarrow H(\pi_{2*}F \otimes \mathbb{C})$ be a map with the properties described above. Then the map*

$$g(p) : E_{\mathcal{D}}(p) \rightarrow F_{\mathcal{D}}(p)$$

is a stable equivalence in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_)$. In particular, the induced homomorphism*

$$g_{\mathcal{D}}(p) : E_{\mathcal{D}}^n(p)(\mathcal{X}) \rightarrow F_{\mathcal{D}}^n(p)(\mathcal{X})$$

is an isomorphism for every $\mathcal{X} \in \mathrm{Sp}^{\Sigma}(\mathbf{sPre}_)$.*

Proof. Since g is a homotopy equivalence, the vertical maps of diagram (15) are pointwise equivalences and hence stable equivalences in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_*)$. Hence the induced map of homotopy pullbacks is a stable equivalence as well. \square

Remark 4.10. The definition of Hodge filtered E -cohomology groups involves the choice of a map

$$\tau^E : E \wedge H\mathbb{C} \rightarrow H(\pi_{2*}E \otimes \mathbb{C}).$$

By our assumption on E , the space of such choices is simply connected. This implies that the Hodge filtered E -cohomology groups of Definition 4.2 are independent of this choice, and a map $g : E \rightarrow F$ of rationally even spectra induces a well defined map from Hodge filtered E -cohomology groups to Hodge filtered F cohomology groups.

4.3. The fundamental short exact sequence for compact Kähler manifolds. From now on we assume that X is a compact complex Kähler manifold. Let p be an integer and let E be a rationally even spectrum. We will show that the group $E_{\mathcal{D}}^{2p}(p)X$ fits into a short exact sequence that is a natural generalization of the short exact sequence for Deligne cohomology (1). The argument is analogous to the case of Deligne cohomology for which we refer to [31, §12].

Therefore, we split the long exact sequence of Proposition 5.3 into a short exact sequence. For a compact Kähler manifold X , the Hodge filtration yields an isomorphism

$$H^{n+2j}(X; \Omega^{*\geq p+j}(\pi_{2j}E \otimes \mathbb{C})) \cong F^{p+j}H^{n+2j}(X; \pi_{2j}E \otimes \mathbb{C}).$$

There is an analogue of the intermediate Jacobian defined which we will denote by $J_E^{2p-1}(X)$. It arises as the cokernel of the map

$$(16) \quad E^{2p-1}X \oplus F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1} \rightarrow H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1}$$

where we use the notation

$$F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1} := \bigoplus F^{p+j}H^{2p-1+2j}(X; \pi_{2j}E \otimes \mathbb{C}).$$

For every j , the Hodge filtration yields a decomposition as a direct sum

$$H^{2p-1+2j}(X; \pi_{2j}E \otimes \mathbb{C}) \cong F^{p+j}H^{2p-1+2j}(X; \pi_{2j}E \otimes \mathbb{C}) \oplus \overline{F^{p+j}H^{2p-1+2j}(X; \pi_{2j}E \otimes \mathbb{C})}$$

where the bar denotes the image under complex conjugation.

Since this is a direct sum decomposition, the intersection

$$F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1} \cap H^*(X; \pi_{2*}E \otimes \mathbb{R})^{2p-1} = \{0\}$$

is trivial. Considering $F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1}$ as a subspace of $E^{2p-1}(X) \otimes \mathbb{C}$ we see that the map

$$E^{2p-1}(X) \otimes \mathbb{R} \rightarrow E^{2p-1}(X) \otimes \mathbb{C} / F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1}$$

is an isomorphism of \mathbb{R} -vector spaces. Therefore, the lattice

$$E^{2p-1}(X) \subset E^{2p-1}(X) \otimes \mathbb{R}$$

is a lattice in the \mathbb{C} -vector space

$$E^{2p-1}(X) \otimes \mathbb{C} / F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1}.$$

Definition 4.11. We define the p th generalized Jacobian of X to be the complex torus

$$J_E^{2p-1}(X) := E^{2p-1}(X) \otimes \mathbb{C} / (F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p-1} \oplus E^{2p-1}(X)).$$

Remark 4.12. The p th generalized Jacobian of X is isomorphic to $E^{2p-1}(X) \otimes \mathbb{R} / \mathbb{Z}$ where the latter group inherits its complex structure from a complex structure on \mathbb{R} / \mathbb{Z} . In particular, $J_E^{2p-1}(X)$ is a homotopy invariant of X .

To complete the picture, let $\text{Hdg}_E^{2p}(X)$ be the subgroup of $E^{2p}(X)$ that is given as the pullback

$$(17) \quad \begin{array}{ccc} \text{Hdg}_E^{2p}(X) & \xrightarrow{\quad\quad\quad} & E^{2p}(X) \\ \downarrow & & \downarrow \\ \bigoplus_j F^{p+j}H^{n+2j}(X; \pi_{2j}E \otimes \mathbb{C}) & \longrightarrow & H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p}. \end{array}$$

The group $\mathrm{Hdg}_E^{2p}(X)$ is determined by the Hodge structure on the cohomology of the Kähler manifold X . The Hodge decomposition of the lower right corner of (17) yields a splitting as a direct sum

$$H^*(X; \pi_{2*}E \otimes \mathbb{C})^{2p} = \bigoplus_{j \geq 0} \left(\bigoplus_{s+t=2(p+j)} H^{s,t}(X; \mathbb{C}) \otimes \pi_{2j}E \right).$$

By Hodge symmetry, we see that $\mathrm{Hdg}_E^{2p}(X)$ is the subgroup of $E^{2p}(X)$ consisting of those elements whose images under the map

$$E^{2p}(X) \rightarrow \bigoplus_j \left(\bigoplus_{s+t=2(p+j)} H^{s,t}(X; \mathbb{C}) \otimes \pi_{2j}E \right)$$

lie in the groups $H^{p+j,p+j}(X; \mathbb{C}) \otimes \pi_{2j}E$ for $j \in \mathbb{Z}$.

Summarizing we obtain the following theorem whose second assertion follows immediately from Theorem 2.13.

Theorem 4.13. *Let p be an integer, X be a compact complex Kähler manifold and let E be a rationally even spectrum. The Hodge filtered complex cohomology groups $E_{\mathcal{D}}^{2p}(p)(X)$ fit into the short exact sequence*

$$(18) \quad 0 \rightarrow J_E^{2p-1}(X) \rightarrow E_{\mathcal{D}}^{2p}(p)(X) \rightarrow \mathrm{Hdg}_E^{2p}(X) \rightarrow 0$$

where $\mathrm{Hdg}_E^{2p}(X)$ is the pullback defined by (17).

Moreover, if there is a map $E \rightarrow H\mathbb{Z}$ of symmetric spectra, we obtain an induced map of short exact sequences

$$(19) \quad \begin{array}{ccccccc} 0 & \longrightarrow & J_E^{2p-1}(X) & \longrightarrow & E_{\mathcal{D}}^{2p}(p)X & \longrightarrow & \mathrm{Hdg}_E^{2p}X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J^{2p-1}(X) & \longrightarrow & H_{\mathcal{D}}^{2p}(X; \mathbb{Z}(p)) & \longrightarrow & \mathrm{Hdg}^{2p}(X; \mathbb{Z}) \longrightarrow 0 \end{array}$$

5. HODGE FILTERED COMPLEX BORDISM

In this section we focus on the case of Hodge filtered complex cohomology groups defined via the Thom spectrum MU of complex bordism. Before we apply the construction let us have a closer look at the choice of the map

$$MU \rightarrow H(\pi_{2*}MU \otimes \mathbb{C}).$$

Although any two choices give the same cohomology groups, we have to take a little more care in order to get a multiplicative structure. We explain this point only for the example of MU and leave it to the reader to pick his favorite spectrum E and to do the necessary adjustments.

For $E = MU$, the three corners of the diagram corresponding to (9) that would define the Hodge filtered cobordism spectrum as a homotopy pullback all carry a natural product structure. These structures will induce a natural product structure on Hodge filtered cobordism groups, once we make sure that the maps are multiplicative, i.e. preserve these product structures. By the construction of monoidal Eilenberg-MacLane spectra, the map

$$H(\Omega^{*\geq p}(\pi_{2*}MU \otimes \mathbb{C})) \rightarrow H(\Omega^*(\pi_{2*}MU \otimes \mathbb{C}))$$

induced by the canonical inclusion of complexes is a morphism of monoids in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)$.

For the other map we make use of the fact that the rationalized spectrum $MU_{\mathbb{Q}} \approx H\mathbb{Q} \wedge MU$ is a free commutative algebra over $H\mathbb{Q}$, and so the commutative \mathbb{S} -algebra $MU \rightarrow E$ to any rational commutative \mathbb{S} -algebra E are in one to one correspondence with the ring homomorphisms $\pi_* MU \rightarrow \pi_* E$. The correspondence is the one associating to a map its effect on homotopy groups. We can therefore specify, up to a choice of point in a simply connected space, a map of commutative \mathbb{S} -algebras

$$\iota : MU \rightarrow H(\pi_* MU \otimes \mathbb{C}),$$

by requiring that for every n the induced map

$$\pi_{2n} \iota : \pi_{2n} MU \rightarrow H(\pi_* MU \otimes \mathbb{C}) = \pi_{2n} MU \otimes \mathbb{C}$$

is multiplication by $(2\pi i)^n$. We similarly obtain a map of commutative \mathbb{S} -algebras

$$\tau' := \bigvee_{p \in \mathbb{Z}} (\tau')^p : \bigvee_{p \in \mathbb{Z}} MU \rightarrow \bigvee_{p \in \mathbb{Z}} H(\pi_* MU \otimes \mathbb{C})$$

in which $\pi_*(\tau')^p = (2\pi i)^p \pi_* \iota$. Composing with the de Rham equivalence

$$\bigvee_{p \in \mathbb{Z}} H(\Omega(\pi_* MU \otimes \mathbb{C})) \approx \bigvee_{p \in \mathbb{Z}} H(\pi_* MU \otimes \mathbb{C})$$

gives a map

$$\tau := \bigvee_{p \in \mathbb{Z}} \tau^p : \bigvee_{p \in \mathbb{Z}} MU \rightarrow \bigvee_{p \in \mathbb{Z}} H(\Omega(\pi_* MU \otimes \mathbb{C}))$$

of commutative \mathbb{S} -algebras in *presheaves of symmetric spectra*, whose stalkwise effect on homotopy groups is

$$\pi_* \tau^p = (2\pi i)^p \pi_* \iota.$$

We now define a presheaf of commutative \mathbb{S} -algebras

$$(20) \quad \bigvee_{p \in \mathbb{Z}} MU_{\mathcal{D}}(p)$$

by the homotopy cartesian square

$$(21) \quad \begin{array}{ccc} \bigvee_{p \in \mathbb{Z}} MU_{\mathcal{D}}(p) & \xrightarrow{\quad} & \bigvee_{p \in \mathbb{Z}} MU \\ \downarrow & & \downarrow \tau \\ \bigvee_{p \in \mathbb{Z}} H(\Omega^{*\geq p}(\pi_* MU \otimes \mathbb{C})) & \xrightarrow{\quad} & \bigvee_{p \in \mathbb{Z}} H(\Omega^*(\pi_* MU \otimes \mathbb{C})) \end{array}$$

and for every presheaf of symmetric spectra \mathcal{X} , and $p \in \mathbb{Z}$ the *Hodge filtered complex bordism groups*

$$MU_{\mathcal{D}}^n(p) := \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)}(\mathcal{X}, \Sigma^n MU_{\mathcal{D}}(p)).$$

This defines in particular, for every complex manifold X , Hodge filtered complex bordism groups $MU_{\mathcal{D}}^n(p)(X)$. Since (20) is a commutative \mathbb{S} -algebra, we get the following multiplicative structure on Hodge filtered complex bordisms groups of complex manifolds.

Theorem 5.1. *For every complex manifold X , there is a multiplicative structure on Hodge filtered complex bordism groups*

$$MU_{\mathcal{D}}^n(p)(X) \otimes MU_{\mathcal{D}}^m(q)(X) \rightarrow MU_{\mathcal{D}}^{n+m}(p+q)(X)$$

induced by the structure of a commutative \mathbb{S} -algebra in $\mathrm{Sp}^{\Sigma}(\mathbf{sPre}_)$ on (20). The multiplication is graded in the sense that for $\alpha \in MU_{\mathcal{D}}^n(p)(X)$, $\beta \in MU_{\mathcal{D}}^m(q)(X)$ we have*

$$\alpha\beta = (-1)^{n+m}\beta\alpha \in MU_{\mathcal{D}}^{n+m}(p+q)(X).$$

Hence there is a structure as a graded commutative ring on

$$MU_{\mathcal{D}}^*(X) := \bigoplus_{n,p} MU_{\mathcal{D}}^n(p)(X).$$

Moreover, if $Y \rightarrow X$ is a holomorphic map of complex manifolds, the induced map of simplicial presheaves $X \rightarrow X/Y$ turns $MU_{\mathcal{D}}^(X)$ into an $MU_{\mathcal{D}}^*(X, Y)$ -module.*

Proposition 5.2. *For every complex manifold X and every n and p , the map of spectra $MU \rightarrow H\mathbb{Z}$ induces a map from Hodge filtered complex bordism to Deligne cohomology groups of X*

$$MU_{\mathcal{D}}^n(p)(X) \rightarrow H_{\mathcal{D}}^n(X; \mathbb{Z}(p))$$

which respects the products structures on both sides.

Proof. This is a consequence of Theorem 2.13 and Theorem 5.1. \square

Proposition 5.3. *For every complex manifold X , Hodge filtered complex bordism groups sit in long exact sequence*

$$\begin{aligned} \dots \rightarrow MU^{n-1}(X) \otimes \mathbb{C} &\rightarrow MU_{\mathcal{D}}(p)^n(X) \rightarrow \\ \rightarrow MU^n(X) \oplus \mathbb{H}^*(X; \Omega^{*\geq p}(\pi_{2*}MU \otimes \mathbb{C}))^n &\rightarrow MU^n(X) \otimes \mathbb{C} \rightarrow \dots \end{aligned}$$

Proof. This follows again from the definition of $MU_{\mathcal{D}}(p)$ as the homotopy pullback in diagram (21) and the isomorphism

$$MU^n(X) \otimes \mathbb{C} \cong H^*(X; \pi_{2*}MU \otimes \mathbb{C})^n.$$

\square

Example 5.4. The Hodge filtered cobordism groups of a point can be read off from the long exact sequence of Proposition 5.3. We split sequence (5.3) into

$$0 \rightarrow \mathrm{coker}(\alpha) \rightarrow MU_{\mathcal{D}}^n(p)(\mathrm{pt}) \rightarrow \ker(\beta) \rightarrow 0$$

where α is the map

$$\alpha : MU^{n-1} \oplus \mathbb{H}^*(\mathrm{pt}; \Omega^{*\geq p}(\pi_{2*}MU \otimes \mathbb{C}))^{n-1} \rightarrow MU^{n-1} \otimes \mathbb{C}$$

and β is the map

$$\beta : MU^n \oplus \mathbb{H}^*(\mathrm{pt}; \Omega^{*\geq p}(\pi_{2*}MU \otimes \mathbb{C}))^n \rightarrow MU^n \otimes \mathbb{C}.$$

First let $n = 2k$ be even. In this case, the group $MU^{2k-1} \otimes \mathbb{C}$ vanishes. Hence the cokernel of α is trivial and $MU_{\mathcal{D}}^n(p)(\mathrm{pt})$ is isomorphic to $\ker(\beta)$. The groups $\mathbb{H}^{n+2j}(\mathrm{pt}; \Omega^{*\geq p+j}(\pi_{2*}MU \otimes \mathbb{C}))$ vanish if $n+2j \neq 0$ or if $p+j > 0$. Hence if $p > 0$ or $k < p$, we see that $MU_{\mathcal{D}}^{2k}(p)(\mathrm{pt})$ is isomorphic to the kernel of the inclusion $MU^{2k} \rightarrow MU^{2k} \otimes \mathbb{C}$ and hence vanishes.

If $p \leq 0$, then $\mathbb{H}^*(\text{pt}; \Omega^{\geq p}(\pi_{2*} MU \otimes \mathbb{C}))^{2k}$ is isomorphic to $MU^{2k} \otimes \mathbb{C}$ for $k \geq p$ and vanishes otherwise. Hence, for $k \geq p$, $MU_{\mathcal{D}}^{2k}(p)(\text{pt})$ is isomorphic to the kernel of the map

$$MU^{2k} \oplus MU^{2k} \otimes \mathbb{C} \rightarrow MU^{2k} \otimes \mathbb{C}$$

and hence

$$MU_{\mathcal{D}}^{2k}(p)(\text{pt}) \cong MU^{2k}$$

for $k \geq p$. In particular, this implies that the group $MU_{\mathcal{D}}^{2k}(p)(\text{pt})$ vanishes for $k > 0$.

We remark that, for all $p \in \mathbb{Z}$, we have

$$MU_{\mathcal{D}}^{2p}(p)(\text{pt}) \cong MU^{2p}.$$

For $n = 2k + 1$ odd, the group $\ker(\beta)$ is trivial, since the source of β vanishes. So $MU_{\mathcal{D}}^{2k+1}(p)(\text{pt})$ is isomorphic to the cokernel of α . For $p > 0$, using the identifications above, we get for all $k \in \mathbb{Z}$

$$MU_{\mathcal{D}}^{2k+1}(p)(\text{pt}) \cong (MU^{2k} \otimes \mathbb{C}) / MU^{2k} \cong MU^{2k} \otimes \mathbb{C} / \mathbb{Z}.$$

In particular, this implies that the group $MU_{\mathcal{D}}^{2k+1}(p)(\text{pt})$ vanishes for $k > 0$.

For $p \leq 0$ and $k < p$, we still get

$$MU_{\mathcal{D}}^{2k+1}(p)(\text{pt}) \cong MU^{2k} \otimes \mathbb{C} / \mathbb{Z}.$$

For $p \leq 0$ and $k \geq p$, we have

$$MU_{\mathcal{D}}^{2k+1}(p)(\text{pt}) \cong (MU^{2k} \otimes \mathbb{C}) / (MU^{2k} \oplus MU^{2k} \otimes \mathbb{C}) = 0.$$

6. HODGE FILTERED COHOMOLOGY FOR COMPLEX ALGEBRAIC VARIETIES

We now turn to the case of algebraic varieties and, as described in the introduction, introduce a variation of our construction which takes into account mixed Hodge structures.

6.1. From the analytic to the Nisnevich topology. To transfer the theory of Hodge filtered cohomology to algebraic varieties, we have to discuss the relationship between the analytic site and the algebraic site. We will use the term *complex variety* for a reduced scheme of finite type over \mathbb{C} . Let $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$ be the site of smooth complex algebraic varieties with the Nisnevich topology.

For $X \in \mathbf{Sm}_{\mathbb{C}, \text{Nis}}$, we denote by X_{an} the associated complex manifold. Together with the analytic topology, X_{an} is an object in $\mathbf{Man}_{\mathbb{C}}$. This defines a functor

$$f^{-1} : \mathbf{Sm}_{\mathbb{C}} \rightarrow \mathbf{Man}_{\mathbb{C}}, \quad X \mapsto f^{-1}(X) := X_{\text{an}}.$$

Composition with f^{-1} induces a functor

$$f_* : \text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Man}_{\mathbb{C}})) \rightarrow \text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \text{Nis}}))$$

of the corresponding categories of presheaves of symmetric spectra. It is the right adjoint of a functor

$$f^* : \text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \text{Nis}})) \rightarrow \text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Man}_{\mathbb{C}})).$$

The pair of functors (f^*, f_*) forms a Quillen pair of adjoints between the model categories of monoids in $\text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \text{Nis}}))$ and $\text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Man}_{\mathbb{C}}))$ respectively.

We let Rf_* be the right derived functor. It is given by the composition $f^{-1} \circ R_{\text{an}}$ of f^{-1} with a fibrant replacement functor R_{an} in the model structure of monoids

in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*(\mathbf{Man}_\mathbb{C}))$. Since representable objects are cofibrant, the adjointness implies the following fact.

Proposition 6.1. *Let X be a smooth complex algebraic variety and let \mathcal{E} be a presheaf of symmetric spectra on the site $\mathbf{Man}_\mathbb{C}$. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*(\mathbf{Man}_\mathbb{C}))}(\Sigma_+^\infty(X_{\mathrm{an}}), \Sigma^n \mathcal{E}) \cong \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}}))}(\Sigma_+^\infty(X), \Sigma^n Rf_* \mathcal{E}).$$

Example 6.2. In particular, if $E\mathrm{Sp}^\Sigma$ is a symmetric spectrum, we have a natural isomorphism

$$E^n(X_{\mathrm{an}}) \cong \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}}))}(\Sigma_+^\infty(X), \Sigma^n Rf_* E).$$

6.2. Differential forms with logarithmic poles. For a smooth complex variety X , by Hironaka's theorem [13], there exists a smooth projective variety \overline{X} over \mathbb{C} with a closed embedding $j : X \hookrightarrow \overline{X}$ such that $D := \overline{X} - X$ is a normal crossing divisor which is the union of smooth divisors. Let $\Omega_{\overline{X}}^1 \langle D \rangle$ be the locally free submodule of $j_* \Omega_X^1$ generated by Ω_X^1 and by $\frac{dz_i}{z_i}$ where z_i is a local equation for an irreducible local component of D . The sheaf $\Omega_{\overline{X}}^p \langle D \rangle$ of meromorphic p -forms on \overline{X} with at most logarithmic poles along D is defined to be the locally free sub-sheaf $\bigwedge^p \Omega_{\overline{X}}^1 \langle D \rangle$ of $j_* \Omega_X^p$. The Hodge filtration F on the complex of sheaves $\Omega_{\overline{X}}^* \langle D \rangle$ is just the naive filtration

$$F^p \Omega_{\overline{X}}^* \langle D \rangle = \Omega_{\overline{X}}^{\geq p} \langle D \rangle.$$

The filtration on the complex of logarithmic forms allows one to define a Hodge filtration on the complex cohomology of X as

$$(22) \quad F^p H^n(X; \mathbb{C}) := \mathrm{Im}(\mathbb{H}^n(\overline{X}; \Omega_{\overline{X}}^{\geq p} \langle D \rangle) \rightarrow \mathbb{H}^n(\overline{X}; \Omega_{\overline{X}}^* \langle D \rangle)) \cong H^n(X; \mathbb{C}),$$

It is a theorem of Deligne's that for smooth complex algebraic varieties, the Hodge filtration is actually detected by the filtration on the complex of logarithmic forms. More precisely, Deligne showed in [5, Corollaire 3.2.13] that the logarithmic Hodge spectral sequence degenerates at the E_1 -level and that the map (22) is an isomorphism

$$(23) \quad F^p H^n(X; \mathbb{C}) \cong \mathbb{H}^n(\overline{X}; \Omega_{\overline{X}}^{\geq p} \langle D \rangle),$$

or in other words, the homomorphism

$$\mathbb{H}^n(\overline{X}; \Omega_{\overline{X}}^{\geq p} \langle D \rangle) \rightarrow \mathbb{H}^n(\overline{X}; \Omega_{\overline{X}}^* \langle D \rangle)$$

of hypercohomology groups is injective.

Following Beilinson [1, §1.6] we now show that the Hodge filtration $F^p H^n(X, \mathbb{C}) \subset H^n(X, \mathbb{C})$ is represented by a map of symmetric spectra. More precisely, we construct a sequence of complexes of presheaves

$$\cdots \rightarrow A_{\log}^p \rightarrow A_{\log}^{p+1} \rightarrow \cdots$$

on $\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}}$ equipped with a weak equivalence

$$\mathrm{colim}_{p \rightarrow \infty} H A_{\log}^p \simeq Rf_* HC,$$

and having the property that for all $X \in \mathbf{Sm}_\mathbb{C}$, the map

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}})}(\Sigma_+^\infty X, \Sigma^n H A_{\log}^p) \rightarrow \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}})}(\Sigma_+^\infty X, \Sigma^n Rf_* HC)$$

induces an isomorphism

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_\mathbb{C}, \mathrm{Nis})}(\Sigma_+^\infty X, \Sigma^n HA_{\log}^p) \rightarrow F^p H^n(X, \mathbb{C}).$$

We consider the category $\overline{\mathbf{Sm}}_\mathbb{C}$ whose objects are *smooth compactifications*, i.e. pairs $X \subset \overline{X}$ consisting of a smooth variety X embedded as an open subset of a projective variety \overline{X} and having the property that $\overline{X} - X$ is a divisor with normal crossings. A map from $X \subset \overline{X}$ to $Y \subset \overline{Y}$ is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \overline{Y}, \end{array}$$

and a collection of maps $\{(U_\alpha \subset \overline{U}_\alpha)\} \rightarrow (X \subset \overline{X})$ is a covering if the maps $U_\alpha \rightarrow X$ form a Nisnevich covering. This defines a Grothendieck topology on $\overline{\mathbf{Sm}}_\mathbb{C}$ and we therefore have Quillen model categories $\mathbf{sPre}(\overline{\mathbf{Sm}}_\mathbb{C})$ of simplicial presheaves on $\overline{\mathbf{Sm}}_\mathbb{C}$ and $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}}_\mathbb{C})$ of presheaves of symmetric spectra. The forgetful functor

$$\begin{aligned} u : \overline{\mathbf{Sm}}_\mathbb{C} &\rightarrow \mathbf{Sm}_\mathbb{C} \\ (X \subset \overline{X}) &\mapsto X \end{aligned}$$

leads to Quillen pairs

$$\begin{aligned} u^* : \mathbf{sPre}(\overline{\mathbf{Sm}}_\mathbb{C}) &\leftrightarrow \mathbf{sPre}(\mathbf{Sm}_\mathbb{C}) : u_* \\ u^* : \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}}_\mathbb{C}) &\leftrightarrow \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_\mathbb{C}) : u_* \end{aligned}$$

and hence adjoint derived functors

$$\begin{aligned} Lu^* : \mathbf{sPre}(\overline{\mathbf{Sm}}_\mathbb{C}) &\leftrightarrow \mathbf{sPre}(\mathbf{Sm}_\mathbb{C}) : Ru_* \\ Lu^* : \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}}_\mathbb{C}) &\leftrightarrow \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_\mathbb{C}) : Ru_*. \end{aligned}$$

The complexes $\Omega_{\overline{X}}^{*\geq p}\langle D \rangle$ form a complex of presheaves $\Omega^{*\geq p}\langle D \rangle$ on $\overline{\mathbf{Sm}}_\mathbb{C}$. Let

$$\Omega_{\overline{X}}^{*\geq p}\langle D \rangle \rightarrow A_{\overline{X}}^p\langle D \rangle$$

be any resolution by cohomologically trivial sheaves which is functorial in \overline{X} , and let

$$\Omega^{*\geq p}\langle D \rangle \rightarrow A^p\langle D \rangle$$

be the associated map of complexes of presheaves. For example, $A^p\langle D \rangle$ could be the Godement resolution, or the logarithmic Dolbeault resolution ([22, §8]). The $A^p\langle D \rangle$ are double complexes, though we will only consider their total complexes. They may be chosen in such a way as to fit into a commutative diagram

$$\begin{array}{ccc} \Omega^{p-1}\langle D \rangle & \longrightarrow & \Omega^p\langle D \rangle \\ \downarrow & & \downarrow \\ A^{p-1}\langle D \rangle & \longrightarrow & A^p\langle D \rangle \end{array}$$

Proposition 6.3. *For every $X \subset \overline{X}$, and every $n \geq 0$, the natural map*

$$\pi(\Sigma_+^\infty(X \subset \overline{X}), \Sigma^n HA^p\langle D \rangle) \rightarrow \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}_\mathbb{C}})}(\Sigma_+^\infty(X \subset \overline{X}), \Sigma^n HA^p\langle D \rangle)$$

is a bijection. Hence

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}_\mathbb{C}})}(\Sigma_+^\infty(X \subset \overline{X}), \Sigma^n HA^p\langle D \rangle) \cong F^p H^n(X; \mathbb{C}).$$

Proof. This follows, as in [4, §5], from the Verdier hypercovering theorem, and the Mayer-Vietoris sequence for the groups $F^p H^n(X, \mathbb{C})$. \square

Finally we define HA_{log}^p to be $Ru_* HA^p\langle D \rangle \in \mathrm{Sp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}})$. By the definition of HA_{log}^p and the fact that representable objects are cofibrant, Proposition 6.3 implies that we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\overline{\mathbf{Sm}_\mathbb{C}})}(\Sigma_+^\infty(X \subset \overline{X}), \Sigma^n HA^p\langle D \rangle) \rightarrow \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}})}(\Sigma_+^\infty X, \Sigma^n HA_{\mathrm{log}}^p).$$

This implies in particular that we obtain an induced isomorphism

$$(24) \quad \mathrm{Hom}_{\mathrm{hoSp}^\Sigma(\mathbf{sPre}_*)(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}})}(\Sigma_+^\infty X, \Sigma^n HA_{\mathrm{log}}^p) \rightarrow F^p H^n(X, \mathbb{C}).$$

6.3. Generalized Deligne-Beilinson cohomology for smooth complex algebraic varieties. Let $A^* = \bigoplus A^{p,q}$ be the presheaf given by the Dolbeault (double) complex of smooth forms with complex coefficients. For an evenly graded \mathbb{C} -algebra \mathcal{V}_{2*} , let $HA^*(\mathcal{V}_{2*})$ denote the presheaf of differential graded \mathbb{C} -algebras

$$HA^*(\mathcal{V}_{2*}) = \bigvee_j \Sigma^{2j} HA^*(\mathcal{V}_{2j}) = \bigvee_j \Sigma^{2j} A^* \otimes \mathcal{V}_{2j}.$$

Moreover, let $HA_{\mathrm{log}}^{p+*}(\mathcal{V}_{2*})$ be the presheaf of symmetric spectra

$$HA_{\mathrm{log}}^{p+*}(\mathcal{V}_{2*}) := \bigvee_j \Sigma^{2j} HA_{\mathrm{log}}^{p+j}(\mathcal{V}_{2j}),$$

where we denote

$$H(A_{\mathrm{log}}^{p+j}(\mathcal{V}_{2j}) := Ru_* H(A^{p+j}\langle D \rangle \otimes \mathcal{V}_{2j}).$$

By choosing the logarithmic Dolbeault complex of [22, §8] as a resolution of the complexes of logarithmic forms for the construction of $A^p\langle D \rangle$, we are equipped with a natural morphism of presheaves of symmetric spectra

$$(25) \quad H(A_{\mathrm{log}}^{p+*}(\mathcal{V}_{2j})) \rightarrow f_* H(A^*(\mathcal{V}_{2j})) \rightarrow Rf_* H(A^*(\mathcal{V}_{2j})).$$

Let E be a rationally even spectrum in Sp^Σ . Let

$$\tau^E : E \rightarrow H(\pi_{2*} E \otimes \mathbb{C})$$

be the map (7). It induces a map

$$Rf_* E \rightarrow Rf_* H(\pi_{2*} E \otimes \mathbb{C})$$

in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}}))$. Composition with the maps

$$Rf_* H(\pi_{2*} E \otimes \mathbb{C}) \rightarrow Rf_* H(A^*(\pi_{2*} E \otimes \mathbb{C}))$$

defines a map in $\mathrm{Sp}^\Sigma(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \mathrm{Nis}}))$

$$Rf_* E \rightarrow Rf_* H(A^*(\pi_{2*} E \otimes \mathbb{C})).$$

We define the presheaf of symmetric spectra $E_{\log}(p)$ by the homotopy cartesian square

$$(26) \quad \begin{array}{ccc} E_{\log}(p) & \xrightarrow{\quad} & Rf_*E \\ \downarrow & & \downarrow \\ H(A_{\log}^{p+*}(\pi_{2*}E \otimes \mathbb{C})) & \longrightarrow & Rf_*H(A^*(\pi_{2*}E \otimes \mathbb{C})). \end{array}$$

Definition 6.4. For a presheaf of symmetric spectra \mathcal{X} on $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$, we define the logarithmic Hodge filtered E -cohomology groups to be

$$E_{\log}(p)^n(\mathcal{X}) := \text{Hom}_{\text{hoSp}^{\Sigma}(\mathbf{sPre}_*)}(\mathcal{X}, \Sigma^n E_{\log}(p)).$$

For a smooth complex algebraic variety X , we denote the E -cohomology group $E^n(X_{\text{an}})$ of the associated complex manifold X_{an} by $E^n(X)$. Moreover, we use the notation

$$F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^n = \bigoplus_j F^{p+j}H^{n+2j}(X; \pi_{2j}E \otimes \mathbb{C})$$

for the sum of Hodge filtered cohomology groups.

Proposition 6.5. *Hodge filtered complex bordism groups sit in long exact sequences*

$$\begin{array}{ccccc} \dots & \rightarrow & H^*(X; \pi_{2*}E \otimes \mathbb{C})^{n-1} & \rightarrow & E_{\log}(p)^n(X) \rightarrow \\ & & \rightarrow E^n(X) \oplus F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^n & \rightarrow & H^*(X; \pi_{2*}E \otimes \mathbb{C})^n \rightarrow \dots \end{array}$$

and

$$\begin{array}{ccccc} \dots & \rightarrow & E^{n-1}(X) \otimes \mathbb{C} & \rightarrow & E_{\log}(p)^n(X) \rightarrow \\ & & \rightarrow E^n(X) \oplus F^{p+*}H^*(X; \pi_{2*}E \otimes \mathbb{C})^n & \rightarrow & E^n(X) \otimes \mathbb{C} \rightarrow \dots \end{array}$$

Proof. As before, this is a consequence of the definition of $E_{\log}(p)$ as a homotopy pullback. The new ingredients are the identifications of E -cohomology groups in Proposition 6.1 and of the Hodge filtered cohomology groups in (24). \square

Remark 6.6. If X is a smooth projective complex variety, the new Hodge filtered cohomology groups $E_{\log}^n(p)(X)$ of Definition 6.4 are canonically isomorphic to the groups $E_{\mathcal{D}}^n(p)(X_{\text{an}})$ of Definition 4.2.

6.4. \mathbb{A}^1 -homotopy invariance.

Theorem 6.7. *Let E be a rationally even ring spectrum and let X be a smooth variety over \mathbb{C} . The projection $\pi : X \times \mathbb{A}^1 \rightarrow X$ induces an isomorphism*

$$\pi^* : E_{\log}^n(p)(X) \xrightarrow{\cong} E_{\log}^n(p)(X \times \mathbb{A}^1).$$

Proof. This can be deduced from the long exact sequence and the \mathbb{A}^1 -invariance of the individual terms. The only non-trivial fact is the \mathbb{A}^1 -invariance of the $F^p H^*(-; \mathbb{C})$ which follows from [5, Théorème 1.2.10]. \square

Proposition 6.8. *Let E be a rationally even ring spectrum. The functor $X \mapsto E_{\log}^*(*)(X)$ on $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$ satisfies Nisnevich descent.*

Proof. It follows from the proof of (24) that the Hodge filtered cohomology satisfies Nisnevich descent, and it is well-known that E -cohomology and de Rham cohomology satisfy Nisnevich descent too. Hence E_{\log} satisfies Nisnevich descent as well. \square

As a consequence of Theorem 6.7 and Proposition 6.8, we get the following result.

Corollary 6.9. *Let E be a rationally even ring spectrum. The functor $E_{\log}^*(*)(-)$ induces a functor on the stable \mathbb{A}^1 -homotopy category of smooth varieties over \mathbb{C} .*

7. GENERALIZED DELIGNE-BEILINSON COBORDISM FOR COMPLEX ALGEBRAIC VARIETIES

In this last section, we apply the refined methods for algebraic complex varieties to the case $E = MU$. The advantage of the logarithmic theory MU_{\log} is that it has several topological properties that would hold for $MU_{\mathcal{D}}$ only for compact manifolds. In particular, we will show that MU_{\log} satisfies a projective bundle formula and is equipped with transfer maps for projective morphisms.

Recall our choice of a map

$$\bigvee_{p \in \mathbb{Z}} MU \rightarrow \bigvee_{p \in \mathbb{Z}} H(\pi_{2*} MU \otimes \mathbb{C})$$

of commutative \mathbb{S} -algebras. We can consider this map as a map of presheaves of commutative \mathbb{S} -algebras on $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$. Composition with the map

$$H(\pi_{2*} MU \otimes \mathbb{C}) \rightarrow H(A^*(\pi_{2*} MU \otimes \mathbb{C}))$$

and application of Rf_* defines a map of presheaves of commutative \mathbb{S} -algebras

$$\bigvee_{p \in \mathbb{Z}} Rf_* MU \rightarrow \bigvee_{p \in \mathbb{Z}} Rf_* H(A^*(\pi_{2*} MU \otimes \mathbb{C})).$$

Moreover, since $\pi_{2*} MU \otimes \mathbb{C}$ is a \mathbb{C} -algebra, we obtain a natural analogue of map (25) as a morphism of presheaves of commutative \mathbb{S} -algebras

$$\bigvee_{p \in \mathbb{Z}} H(A_{\log}^{p+*}(\pi_{2*} MU \otimes \mathbb{C})) \rightarrow \bigvee_{p \in \mathbb{Z}} Rf_* H(A^*(\pi_{2*} MU \otimes \mathbb{C})).$$

We define the presheaf of commutative \mathbb{S} -algebras in $\text{Sp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \text{Nis}}))$

$$(27) \quad \bigvee_{p \in \mathbb{Z}} MU_{\log}(p)$$

by the homotopy cartesian square

$$(28) \quad \begin{array}{ccc} \bigvee_{p \in \mathbb{Z}} MU_{\log}(p) & \xrightarrow{\quad} & \bigvee_{p \in \mathbb{Z}} Rf_* MU \\ \downarrow & & \downarrow \\ \bigvee_{p \in \mathbb{Z}} H(A_{\log}^{p+*}(\pi_{2*} MU \otimes \mathbb{C})) & \xrightarrow{\quad} & \bigvee_{p \in \mathbb{Z}} Rf_* H(A^*(\pi_{2*} MU \otimes \mathbb{C})) \end{array}$$

Definition 7.1. For a presheaf of symmetric spectra \mathcal{X} on $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$, we define the logarithmic Hodge filtered complex bordism groups to be

$$MU_{\log}(p)^n(\mathcal{X}) := \text{Hom}_{\text{hoSp}^{\Sigma}(\mathbf{sPre}_*(\mathbf{Sm}_{\mathbb{C}, \text{Nis}}))}(\mathcal{X}, \Sigma^n MU_{\log}(p)).$$

Moreover, since (27) is a presheaf of commutative \mathbb{S} -algebras, there is a multiplicative structure on $MU_{\log}^*(*)$.

Theorem 7.2. *Let X be a smooth complex variety. There is a multiplication*

$$MU_{\log}^n(p)(X) \otimes MU_{\log}^m(q)(X) \rightarrow MU_{\log}^{n+m}(p+q)(X)$$

which is graded in the sense that for $\alpha \in MU_{\log}^n(p)(X)$, $\beta \in MU_{\log}^m(q)(X)$ we have

$$\alpha\beta = (-1)^{n+m}\beta\alpha \in MU_{\log}^{n+m}(p+q)(X).$$

This provides

$$MU_{\log}^*(*)(X) := \bigoplus_{n,p} MU_{\log}^n(p)(X)$$

with the structure of a graded-commutative ring.

Remark 7.3. For a smooth complex variety X , the map $MU \rightarrow H\mathbb{Z}$ induces a homomorphism from logarithmic Hodge filtered cobordism to Deligne-Beilinson cohomology which is compatible with the product structures.

7.1. Projective bundle formula. Let X be a smooth variety over \mathbb{C} . We would like to define Chern classes

$$c_p(V) = c_p^{MU_{\log}}(V) \in MU_{\log}^{2p}(p)(X)$$

for vector bundles $V \rightarrow X$ of rank r over X . These classes should only depend on the isomorphism class of E , be functorial with respect to pullbacks and compatible with the Chern classes in the topological theory $MU^{2*}(X)$, i.e. $c_p^{MU_{\log}}(V)$ should be mapped to the topological Chern classes of V under the map

$$MU_{\log}^{2p}(p)(X) \rightarrow MU^{2p}(X).$$

To define Chern classes we follow the outline of Beilinson in [1, §1.7] for Deligne cohomology together with the general theory of classifying spaces in the homotopy category of simplicial presheaves on $\mathbf{Sm}_{\mathbb{C}, \text{Nis}}$ in [21, §4.1]. The following result is a consequence of [21, Proposition 4.1.15].

Proposition 7.4. *Let X be a smooth complex variety. The set of isomorphism classes of vector bundles of rank r on X is in bijection with the set of maps in the homotopy category of simplicial presheaves from X to BGL_r . The correspondence sends a map $f : X \rightarrow BGL_r$ to the isomorphism class of the bundle f^*EGL_r of the pullback of the universal bundle EGL_r over BGL_r .*

The second ingredient is the following fact that the Hodge filtered cobordism of the classifying space $BGL_r(\mathbb{C})$ is just given by complex cobordism.

Lemma 7.5. *For every $p \geq 0$, the homomorphism*

$$MU_{\log}^{2p}(p)(BGL_r) \rightarrow MU^{2p}(BGL_r(\mathbb{C}))$$

induced by the map of presheaves of symmetric spectra $MU_{\log}(p) \rightarrow MU$ is an isomorphism.

Proof. This follows from Proposition 6.5 and the following two facts. On the one hand, the odd dimensional cohomology of $BGL_r(\mathbb{C})$ vanishes, i.e.

$$H^{2p-1+2j}(BGL_r(\mathbb{C}); \pi_{2j}MU \otimes \mathbb{C}) = 0.$$

On the other hand, the even cohomology of $BGL_r(\mathbb{C})$ is generated by Chern classes which have pure weight (p, p) (see [6]). This implies

$$\mathbb{H}^{2p+2j}(BGL_r(\mathbb{C}); \Omega^{*\geq p+j}(\pi_{2j}MU \otimes \mathbb{C})) \cong H^{2p+2j}(BGL_r(\mathbb{C}); \pi_{2j}MU \otimes \mathbb{C}).$$

Now one can read off the assertion from the first long exact sequence in Proposition 6.5. \square

Now let V be a vector bundle of rank r over X . By Proposition 7.4, there is a unique map in the homotopy category of simplicial presheaves $f : X \rightarrow BGL_r$ corresponding to V over X . Hence we obtain a pullback map

$$MU_{\log}^*(*)(BGL_r) \rightarrow MU_{\log}^*(*)(X)$$

corresponding to V . The isomorphism of Lemma 7.5 and the Chern classes of the universal bundle over BGL_r in complex cobordism provide Chern classes of the universal bundle in $\bigoplus_p MU_{\log}^{2p}(p)(BGL_r)$. The above pullback map then defines unique Chern classes $c_p(V) \in MU_{\log}(p)^{2p}(X)$. This uniqueness implies that these Chern classes satisfy all the properties they satisfy in topological cobordism. Moreover, there is a projective bundle formula.

Theorem 7.6. *Let X be a smooth complex variety and V a vector bundle of rank r over X with projective bundle $\mathbb{P}(V) \rightarrow X$ and tautological quotient line bundle $\mathcal{O}_V(-1)$. Let $\xi = c_1(\mathcal{O}_V(-1))$ be the first Chern class of $\mathcal{O}_V(-1)$ in $MU_{\log}^2(1)(\mathbb{P}(V))$. There are natural isomorphisms*

$$\bigoplus_{i=0}^{r-1} \xi^i MU_{\log}^{n-2i}(p-i)(X) \rightarrow MU_{\log}^n(p)(\mathbb{P}(V))$$

which make $MU_{\log}^*(*)(\mathbb{P}(V)) := \bigoplus_{p,n} MU_{\log}^n(p)(\mathbb{P}(V))$ into a free $MU_{\log}^*(*)(X)$ -module with basis $1, \xi, \dots, \xi^{r-1}$.

Proof. This follows from Proposition 6.5. For each of the groups in the long exact sequence we have isomorphisms

$$\begin{aligned} MU^{n-1}(\mathbb{P}(V)) &\cong \bigoplus_{0 \leq i \leq r-1} \xi_{MU}^i MU^{n-1-2i}(X), \\ H^*(\mathbb{P}(V); \mathcal{V}_{2*})^n &\cong \bigoplus_{0 \leq i \leq r-1} \xi_H^i H^*(X; \mathcal{V}_{2*})^{n-2i}, \\ F^{p+*} H^*(\mathbb{P}(V); \mathcal{V}_{2*})^n &\cong \bigoplus_{0 \leq i \leq r-1} \xi_H^i F^{p-i+*} H^*(X; \mathcal{V}_{2*})^{2p-2i}. \end{aligned}$$

Since the choices of the bases elements are compatible and since the above long exact sequence is compatible with multiplication and direct sums, we deduce that each $MU_{\log}(p)^n(\mathbb{P}(V))$ is a free $MU_{\log}^{2*}(*)(X)$ -module with basis $1, \xi, \dots, \xi^{r-1}$. \square

Remark 7.7. The Cartan formula implies the Grothendieck formula

$$\sum_{p=0}^r (-1)^p c_p(V) \xi_{MU_{\log}}^{r-p} = 0$$

for Chern classes $c_p(V) \in MU_{\log}^{2p}(p)(X)$.

Proposition 7.8. *Chern classes in Hodge filtered complex cobordism are compatible with Chern classes in complex cobordism and in Deligne cohomology and they are functorial under pullbacks, i.e. for any morphism $f : X \rightarrow Y$ of smooth projective complex schemes, one has*

$$f^* c_p(V) = c_p(f^* V).$$

7.2. Transfer maps. Let $f : Y \rightarrow X$ be a projective morphism between smooth quasi-projective complex varieties of relative dimension d . In this section we will define a transfer or pushforward map

$$f_* : MU_{\log}^*(*)(Y) \rightarrow MU_{\log}^{*+2d}(*+d)(X).$$

In order to construct f_* , we apply the machinery developed by Panin in [23] and [24]. For the functor

$$X \mapsto MU_{\log}^*(*)(X) := \bigoplus_{n,p} MU_{\log}^n(p)(X)$$

defines an oriented ring cohomology theory on the category of smooth quasi-projective complex varieties in the sense of [23, §2]. The existence of an exact localization sequence follows as in Proposition 4.7. The excision property follows from the fact that $MU_{\log}^n(p)(-)$ satisfies Nisnevich descent as shown in Proposition 6.8. The \mathbb{A}^1 -invariance is the content of Theorem 6.7. This shows that the properties of a cohomology theory in the sense of [23, Definition 2.1] are satisfied. The fact that $MU_{\log}^*(*)(-)$ is a ring cohomology theory follows from Theorem 7.2. Finally, the projective bundle formula of Theorem 7.6 yields a Chern structure and hence an orientation on MU_{\log} .

Then by [24, Theorem 2.5] and [23, Theorem 3.35], given a projective morphism $f : Y \rightarrow X$ between smooth quasi-projective complex varieties of relative dimension d , there is a homomorphism

$$f_* : MU_{\log}^*(*)(Y) \rightarrow MU_{\log}^{*+2d}(*+d)(X).$$

The construction of f_* is compatible with the Chern structure. In particular, for a smooth divisor

$$i : D \hookrightarrow X,$$

the pushforward i_* satisfies

$$i_*(1) = c_1(\mathcal{O}_X(D)).$$

Let

$$MU_{\log}^{2*}(*)(X) := \bigoplus_p MU_{\log}^{2p}(p)(X)$$

be the sum of the diagonal Hodge filtered cobordism groups. The existence of the transfer structure implies the following result.

Proposition 7.9. *For every smooth quasi-projective complex scheme X , there is a natural ring homomorphism*

$$\Omega^*(X) \rightarrow MU_{\log}^{2*}(*)(X)$$

from algebraic to Hodge filtered cobordism.

Proof. By [24], the pushforward provides the functor $X \mapsto MU_{\log}^{2*}(*)(X)$ with the structure of an oriented Borel-Moore cohomology theory on the category of smooth quasi-projective complex varieties in the sense of [20]. Hence, by [20], sending the class $[Y \rightarrow X]$ of a projective morphism $f : Y \rightarrow X$ of relative dimension d , with Y smooth and quasi-projective over \mathbb{C} , to $f_*(1_Y) \in MU_{\log}^{2d}(d)(X)$, defines a unique morphism of oriented Borel-Moore cohomology theories

$$\Omega^*(X) \rightarrow MU_{\log}^{2*}(*)(X).$$

□

Remark 7.10. The construction of Chern classes and of the transfer map do not depend on the fact that we work with the Thom spectrum MU . The required data would be a rationally even ring spectrum $E \in \mathbf{Sp}^\Sigma$ representing an oriented cohomology theory, together with a choice of ring map

$$E \wedge HC \rightarrow H(\pi_{2*}E \otimes \mathbb{C})$$

as for the case of MU in (27).

Any such data yields an oriented cohomology theory $E_{\log}^*(*)$ on $\mathbf{Sm}_{\mathbb{C}}$. It is equipped with transfer maps

$$E_{\log}^*(*)(Y) \rightarrow E_{\log}^{*+2d}(*+d)(X)$$

for any projective morphism $Y \rightarrow X$ of relative dimension d between smooth quasi-projective varieties. Moreover, this transfer structure induces a natural map

$$\Omega^*(X) \rightarrow E_{\log}^{2*}(*)(X)$$

for every $X \in \mathbf{Sm}_{\mathbb{C}}$.

The only reason why we worked out the example of $E = MU$ is that we showed that there exists a choice of multiplicative map

$$MU \wedge HC \rightarrow H(\pi_{2*}MU \otimes \mathbb{C})$$

and that we are interested in applications based on complex cobordism. But there are of course plenty of other interesting examples.

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DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA
E-mail address: `mjh@math.harvard.edu`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, CAMBRIDGE, MA 02138, USA
E-mail address: `gquick@math.harvard.edu`